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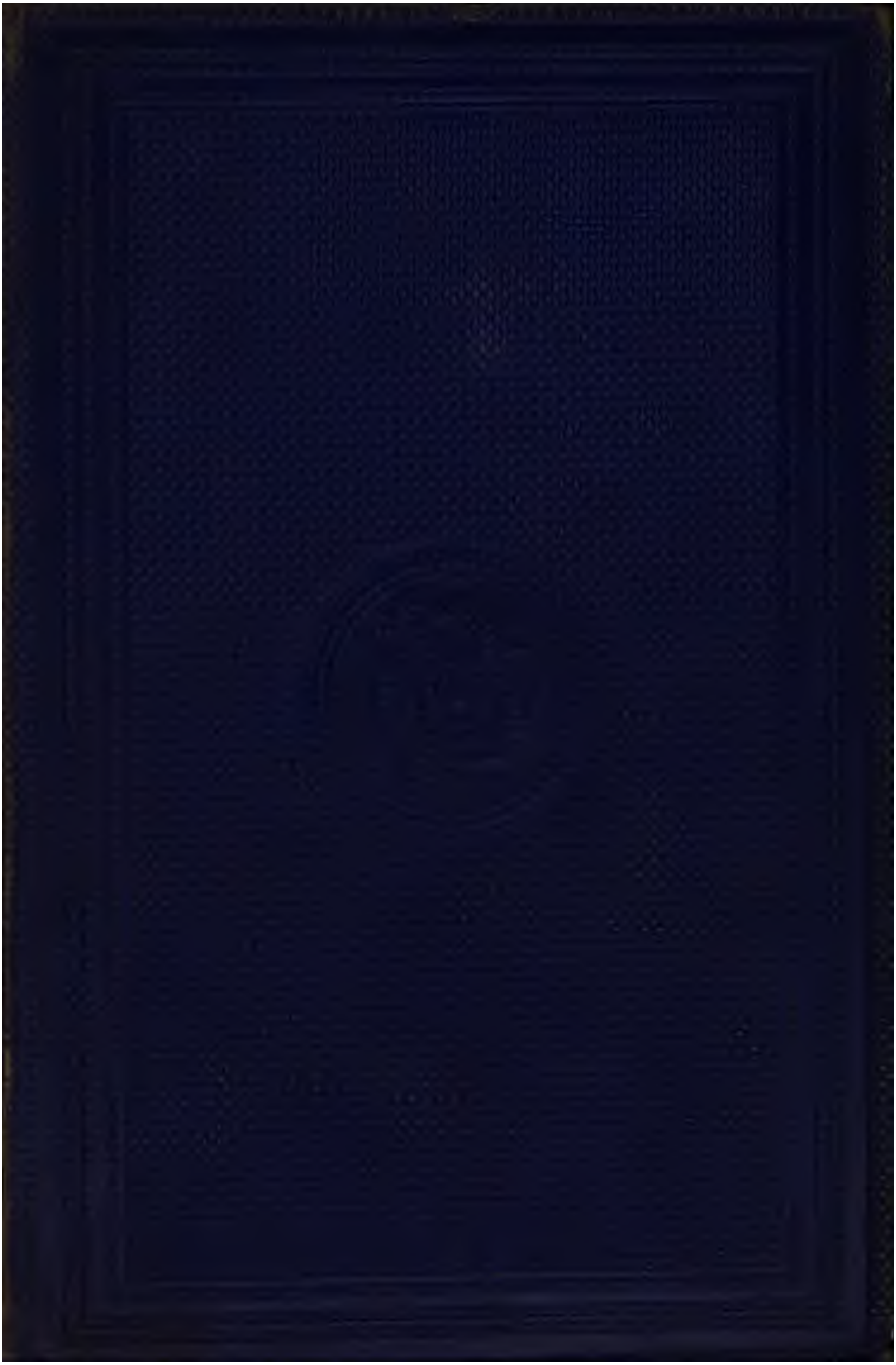

A

TREATISE ON ALGEBRA

BY

JAMES BRYCE, M.A., LL.D., F.G.S.

THIRD EDITION, GREATLY ENLARGED.

~~180. b. 8.~~

EDINBURGH
ADAM AND CHARLES BLACK
1861

181. b. 24.

EDINBURGH: PRINTED BY R. AND R. CLARK.

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PREFACE.

THIS work was originally prepared with the view of combining a greater amount of theory and practice than any elementary treatise which had before appeared. The theory of every operation was fully given, and a very copious collection of exercises was subjoined to each Rule. Great pains were taken to express the Definitions and Rules in the most simple and yet strictly accurate manner, so that their meaning should at once appear, and that the student should from the first be accustomed to the most rigid mathematical exactness.

Into the present edition many important improvements have been introduced. By adopting a more compact arrangement of the matter, and placing the answers together at the end, room has been found for several additional chapters, and for an enlargement of the collection of exercises, chiefly by annexing to each series a few questions of greater difficulty than those given in the former editions. A great many other alterations and improvements have been introduced through-out the work, which cannot be particularly specified; but which, with those already mentioned, will, it is hoped, render it more acceptable to students of mathematics, and fully adapt it to the present advanced state of analytical science.

Every intelligent teacher must be aware that it is of great importance to introduce equations, and questions producing equations, at an early period of the student's progress. The exercises in simple equations are so arranged, that a few of them may be used after the student is acquainted with addition and subtraction; a few more after he has learned division—the rest will require a knowledge of fractions and surds. In a regular treatise, it is necessary to bring together all that relates to the same subject, in whatever order the subjects themselves may be taken up; but, in the practice of teaching, it is not desirable to adhere strictly to this arrangement.

The suggestions offered by several eminent teachers, for the improvement of the work, deserve the author's best thanks; and he will gratefully receive any hints for further improvement, or notices of errors or omissions.

FEBRUARY 1, 1861.

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A
TREATISE ON ALGEBRA.

SECTION I.

DEFINITIONS AND AXIOMS.

ARTICLE 1. The sign + (read *plus*) indicates that the quantity* before which it stands is to be added to something going before: thus, 8 + 6 means that 6 is to be added to 8.

When the sign + is prefixed to a quantity standing alone, or at the beginning of a series, it indicates that there are circumstances which require that quantity to be added, if there were any quantity going before to which it could be joined: thus, + 7 indicates that there are circumstances which require 7 to be added, were there any thing to which the addition could be made.

2. The sign – (read *minus*) indicates that the quantity to which it is prefixed is to be subtracted from something going before: thus 8 – 6 means that 6 is to be taken from 8.

When the sign – is prefixed to a quantity standing alone, or at the beginning of a series, it indicates that there are circumstances which require that quantity to be subtracted, if there were any quantity going before it: thus, -9 signifies that there are circumstances which require 9 to be subtracted, if there were any quantity from which it could be taken.

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* Writers on Algebra generally confine the word *number* to those numerical expressions in which the Arabic figures alone occur; while they apply the term *quantity* to any symbol representing a number, whether consisting of figures, or of letters, or of both combined: thus, 5, 19, are called *numbers*, while 5, 19, 4*a*, may all be called *quantities*.—*See Note at the end of the Volume.*

3. A quantity which has the sign + is called a *positive quantity;* and one which has the sign − is called a *negative quantity.* Quantities to which no sign is prefixed are understood to be positive.

4. The mark = placed between two expressions indicates that they are equal: thus, $12+8=20$, indicates that the sum of 12 and 8 is equal to 20; and $15-9=6$, means that the difference of 15 and 9 is equal to 6; and $19-20=-1$, indicates, that if it be required to take away 20 from 19, the thing is manifestly impossible, and that after we have taken away as much as possible, namely 19, there will still remain a unit to be subtracted, without any thing from which it can be taken. Also, $a+b=c$, shows that some number, which we call a, when increased by the addition of some other number, which we call b, becomes equal to a third number, which we denote by c.

5. The example $19-20=-1$, will help the learner to conceive how a negative quantity may come to stand alone, that is, unconnected with any thing from which it can be taken. It will also show, that, in such a case, the sign − demands that the quantity before which it is placed be subtracted from 0;* that is to say, $-1=0-1$. And since $a-1$, or in general $a-b$, must be less than a, it has been said that -1 or $-b$, that is, $0-1$ or $0-b$, must be less than

---

* For when, of the 20 to be subtracted, we have taken away as much as possible, namely 19, the $+19$ will be reduced to 0, and the -20 to -1.

0. This mode of speaking would be unintelligible and absurd, if used *absolutely;* but in the *relative* sense in which it is always used in Algebra, it is perfectly rational and correct. Suppose the question asked, "How far is Paris north of London?" we may reply, "Paris is $2\frac{1}{2}$ degrees south of London:" but we have not given a direct answer to the question. We have, *first,* rectified the mistake on which the question was founded, by showing that it ought to have been, "How far is Paris south of London?" and, *secondly,* we have given the answer to this corrected question. If we were compelled to answer in exact accordance with the original question, "How far is Paris north of London?" we should say, $2\frac{1}{2}$ degrees less than nothing, that is, $-2\frac{1}{2}$ degrees. This does not mean that there exists, or can exist, any portion of space less than nothing (for that would be a contradiction); but that the *northing* of Paris from London is less than 0: in other words, that not only is Paris nothing to the north of London, but its situation is even less northerly than those words would imply. Therefore the answer, "Paris is $-2\frac{1}{2}$ degrees to the north of London," does not merely express the *absolute* difference of latitude between the two places; but indicates, besides, that their true *relative* position is the very contrary of that which was supposed in the question. The question implies a supposition that the relation of the two places is such, that in order to find the latitude of Paris something must be added to the latitude of London: the answer exhibits this *quantity to be added* as *less than nothing,* which indicates the real relation of the two places to be such, that to find the latitude of Paris something must be subtracted from the latitude of London. In like manner, if it be asked, "How much is Edinburgh south of London?" the *direct* answer will be $-4\frac{1}{2}$ degrees; that is to say, the question supposes the relation of the two places to be such, that in order to find the latitude of Edinburgh something must

be subtracted from that of London: the answer exhibits *this quantity to be subtracted* as *less than nothing*, which indicates the real relation of the two places to be such, that to find the latitude of Edinburgh something must be added to that of London.

6. Hence, to *add* a *negative quantity* is the same as to *subtract that quantity considered absolutely:* for to speak of $-a$ being added to x, is to represent x as increased by a quantity, which is less than 0 by the quantity a. This does not mean that any quantity absolutely less than 0 can exist, but merely that the increase received by x is less than 0: in other words, this mode of expression shows that such a relation had been supposed to exist between some two things concerned in the question, as would have required a to be added; whereas the real relation has turned out to be such as requires a to be subtracted. Also, to *subtract* a *negative quantity* is the same as to *add that quantity considered absolutely.* For, to speak of $-a$ being subtracted from x is to represent x as diminished by a quantity, which is less than 0 by the quantity a: and this indicates that such a relation had been supposed to exist between some two things concerned in the question as would have required a to be subtracted; whereas the real relation has turned out to be one which requires a to be added.

7. When there is a series of quantities connected by the signs +, −, it is of no consequence in what order they are taken: thus, $7+8-3+5-9$, and $8-3+5-9+7$, and $-3+8+5-9+7$, all give the same result, namely 8.

8. The mark × or a dot (.) placed between two quantities indicates that one of them is to be multiplied by the other: thus, 5×7, or $5 . 7$, means that 5 and 7 are to be

multiplied together; and $a \times x$, or $a . x$, that the numbers denoted by a and x, whatever they may be, are to be multiplied together. But when the numbers are not both expressed by the common Arabic figures, it is more usual to indicate their multiplication by writing them close together, like the letters of a word: thus, $5x$ and by are more usually written than $5 \times x$ or $5 . x$, and $b \times y$ or $b . y$.

9. When three or more quantities are connected together by the mark of multiplication, the expression indicates that the first is to be multiplied by the second, their product by the third, that product again by the fourth, and so on: this is called *continual multiplication:* thus $8 \times 6 \times 3$, or $8 . 6 . 3$, signifies that 8 is to be multiplied by 6, and that the product 48 is then to be multiplied by 3, giving for the final product 144. In like manner, $4 \times 7 \times 5 \times 2 = 280$, and $3 \times 3 \times 3 \times 3 \times 3 \times 3 = 729$. Also, $a \times b \times c$, or abc, indicates that a is to be multiplied by b, and that their product is to be again multiplied by c.

10. The quantities by whose multiplication a product is obtained are called its *factors:* thus 5 and 7 are the factors of 35; thus also, the factors of 30 are 6 and 5, or 10 and 3, or 5, 3, and 2; and the product abx may be considered as formed by the three factors, a, and b, and x, or by the two, ab and x.

11. The order in which the factors are taken does not affect the product: thus $5 \times 3 \times 7$, $7 \times 3 \times 5$, $5 \times 7 \times 3$, &c., all give the same result: also, $4abx$, $4xab$, $4bax$, $b4xa$, &c., are all of the same value.* But it is customary to place those

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* Obvious as this proposition may appear, the most ingenious mathematicians have not been able to devise a proof of it sufficiently strict and scientific, and at the same time sufficiently simple to be of any use to the beginner.—*See Note to Art.* 36.

factors first which are expressed in figures, and next those which are denoted by letters, arranging them in the order of the alphabet.

12. When a quantity is considered as the product of two factors, any one of these is called the *coefficient* of the other: thus, in the product $5ax$, 5 is the coefficient of ax, and $5a$ is the coefficient of x. The name *coefficient*, however, is most frequently given to a factor expressed in common Arabic figures, the other factor being a letter or letters: such, for the sake of distinction, are sometimes called *numeral coefficients*. And since a may be considered as the product of 1 and a, ab as the product of 1 and ab, &c., it is plain that 1 must be understood as the coefficient of any quantity expressed by letters alone, whenever it is necessary to speak of such a quantity as having a numeral coefficient; that is to say $a = 1a$, $ab = 1ab$, &c.

13. The mark $\div$ placed between two quantities indicates that the former is to be divided by the latter. Division is also indicated by placing the dividend directly over the divisor, and separating them by a line: thus, to show that 48 is to be divided by 6, we may write either $48 \div 6$, or $\frac{48}{6}$; also, either of the expressions $a \div b$ or $\frac{a}{b}$ denotes that a is to be divided by b.

14. A *power* of a quantity is the product of two or more factors, each equal to that quantity. The powers are distinguished from one another by the number of factors they contain; that is, when there are *two* factors, the product is called the *second power;* when *three*, the *third power;* when *four*, the *fourth power;* and so on. The quantity itself from which the powers are formed, is often, for the sake of

uniformity, called the *first power;* the second power is also called the *square*, and the third power the *cube:*[*] thus,

The 1st power of 3 is 3;
" 2d " 3 " $3 \times 3 = 9$;
" 3d " 3 " $3 \times 3 \times 3 = 27$;
" 4th " 3 " $3 \times 3 \times 3 \times 3 = 81$;
" 5th " 3 " $3 \times 3 \times 3 \times 3 \times 3 = 243$; &c.

In like manner, the 10th power of $5 = 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 9765625$: also, xxx is the third power of x, $yyyyy$ the fifth power of y, $bbbbbbb$ the seventh power of b, and $aaaaaaaa$ the eighth power of a. These examples show that the powers of a letter will be expressed by writing that letter as often as there are units in the number by which the power is distinguished; that is, *twice* for the second power, *three times* for the third power, &c. But as this method would be very inconvenient for the higher powers, it has been agreed not to write the letter more than once, but to place over it and towards the right hand a number indicating how often it ought to be written; that is, how often it is to be used as a factor. Thus, instead of writing xxx for the third power of x, we write x^3, where the number 3 indicates that we are to understand x as written three times; or, which is the same thing, that we are to conceive of a product formed by three factors, each equal to x. A number thus employed to indicate how many factors a power contains is called the *index* or *exponent* of the power. Accordingly —

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[*] The application of the terms *square* and *cube* to algebraic quantities is very improper: it has arisen from the manner in which geometrical magnitudes are expressed in numbers; for whatever number expresses the linear units in a given straight line, the second power of that number will express the number of corresponding superficial units in the area of the square of that line, and its third power the number of solid units in the cube of that line.

The 1st power of a is $a = a^1$, and its index is 1;
„ 2d „ a „ $aa = a^2$, „ 2;
„ 3d „ a „ $aaa = a^3$, „ 3;
„ 4th „ a „ $aaaa = a^4$, „ 4;
„ 5th „ a „ $aaaaa = a^5$, „ 5; &c.

Indices are chiefly used with letters, but sometimes also with common numbers; thus $3^4 = 81$, $4^3 = 64$.

The process of finding any power of a quantity is called *involution;* and the quantity is said to be *raised to* that power.

15. A *root* of a quantity is some other quantity of which the given quantity is a power; thus $8^2 = 64$, therefore 8 is a root of 64; $4^3 = 64$, therefore 4 is also a root of 64; $2^6 = 64$, therefore 2 is another root of 64.

The different roots of a quantity are distinguished by numbers, indicating the power to which each root must be raised, in order to become equal to the given quantity; that is, the root of which the given quantity is the second power is called the *second root,* that of which it is the third power is called the *third root,* that of which it is the fourth power is called the *fourth root,* &c. Thus, in the above example, the second root of 64 is 8, the third root 4, and the sixth root 2. In like manner, the fourth root of 81 is 3, the fifth root of 1024 is 4, &c.

The process of finding any root of a quantity is called *evolution;* and the quantity is said to have that root *extracted.*

The second root is also called the *square* root, and the third the *cube* root (Art. 14.)

16. The mark $\surd$ prefixed to a quantity indicates that a root of that quantity is to be extracted: this mark is called the *radical sign.* The number by which the root is dis-

tinguished is written over the radical sign to the left, and is called the *index of the root:* thus, $\sqrt[2]{25}$ means the second root of 25; $\sqrt[3]{27}$, the third root of 27. In using the radical sign to denote the second root, the index 2 is usually omitted: thus, for the second root of 25, we may either write $\sqrt[2]{25}$, or $\sqrt{25}$. The roots of any number, a, will be expressed as follows:—

The 2d root of a is $\sqrt[2]{a}$, or $\sqrt{a}$.
„ 3d „ a „ $\sqrt[3]{a}$.
„ 4th „ a „ $\sqrt[4]{a}$.
„ 5th „ a „ $\sqrt[5]{a}$, &c.

17. Whatever number of factors a quantity contains besides its coefficient, of so many *dimensions* it is said to be: thus, bc, $7ax$, $3x^2$, are quantities of two dimensions, because each of them contains two factors in addition to its coefficient. In like manner, xyz, $5ax^2$, are quantities of three dimensions; xy^3, $12a^2x^2$, are quantities of four dimensions, and so on. On the same principle, a, $15x$, are quantities of one dimension.

Instead of "one dimension," "two dimensions," &c., the terms *first order* or *degree*, *second order* or *degree*, &c., are often used: thus, $7ax$ is said to be of the *second order* or *second degree;* $5ax^2$, of the *third order* or *third degree*, &c. Sometimes a letter is considered as belonging to the coefficient, and then it is not reckoned in estimating the dimensions or degree of the quantity: thus, in the example $5ax^2$, if we consider $5a$ as the coefficient, the quantity will be of two dimensions only.

18. An expression which does not consist of parts connected by either of the signs + or −, is called a *simple* or *mononomial quantity:* thus, 7 and x are simple quantities. In like manner, $6 \times 3 \times 4$ is a simple quantity: also, $12ax$, and $\frac{bc}{x}$.

19. An expression which consists of two or more simple quantities connected by the signs +, −, is called a *compound quantity*, and the simple quantities of which it is composed are called its *terms:* thus $6 + 3 + 4$ is a compound quantity, and its terms are 6, 3, and 4 : also, $12 + a - x$, and $3y + 12bc - \frac{bx}{y}$, are compound quantities ; the terms of the former being 12, a, and x, and those of the latter, $3y$, $12bc$, and $\frac{bx}{y}$. An expression is said to be *homogeneous* when all its terms are of the same dimensions.

20. A compound quantity consisting of two terms is called a *binomial.* A quantity consisting of more terms than two is called a *multinomial* or *polynomial* quantity. The names *trinomial* and *quadrinomial* are sometimes given to quantities consisting of three and four terms.

21. A parenthesis enclosing an expression is called a *vinculum*, and denotes that the sign or index on the outside of it affects the result of all the operations indicated by the several signs and indices within it. Thus, $18 - (5 + 7)$ means that 7 is first to be added to 5, and then the sum taken from 18, which leaves 6 ; without the vinculum, $18 - 5 + 7$ would mean, that 5 is to be taken from 18, and 7 added to the remainder, which would give 20 : also, $16 - (15 - 9) = 10$, but $16 - 15 - 9 = -8$. In like manner, $7\ (12 + 5)$ or $(12 + 5) \times 7 = 119$, but $12 + 5 \times 7 = 47$. And $(10 + 3)^2 = 169$, but $10 + 3^2 = 19$; and $\sqrt{}(25 + 11) = 6$, but $\sqrt{}25 + 11 = 16$. Also, $(xy)^2$ would mean that x is to be multiplied by y, and the second power of the product taken ; whereas xy^2 would mean that the second power of y was to be taken, and x multiplied by it ; or, supposing $x = 5$ and $y = 3$, $(5 \times 3)^2 = 225$, but $5 \times 3^2 = 45$.

The learner must carefully observe, that in all such cases the sign on the outside of the vinculum affects no one of the terms within the vinculum, each of which has, besides, a sign of its own; that is, in the expressions $18-(5+7)$, and $16-(15-9)$, the terms 5 and 15 are understood to have the sign + (Art. 3).

A bar drawn over the quantities is also used as a vinculum: the bar is sometimes placed vertically: thus—

$\overline{a+b-c}\times d$, and $\begin{array}{r|l} a & d \\ +b & \\ -c & \end{array}$, mean the same as $(a+b-c)\times d$.

Brackets [ ], and the brace { }, are used instead of the parenthesis when we have occasion to place one vinculum outside of another; as, $[(a+b)x+(a-b)x]^2$.

22. *Like quantities* are those in which the parts expressed by letters are the same: thus, $54ay$, $6ay$, and $-19ay$, are like quantities. But $7ay$, and $7a^2y$, are *unlike;* because the factor a occurs twice in the latter, and only once in the former, for (Art. 14) $7a^2y=7aay$. In the same manner $17a^2x$, and $5xa^2$, are like quantities (Art. 11); but $4a^2x$, and $6a^3x$, are unlike.

23. When two or more like quantities are connected by the signs +, −, they may be incorporated into one term, by performing on their coefficients the operations indicated by their signs, and to the result subjoining the common letter: thus, $5x+3x+7x=15x$: for the first term contains five times x to be added, the next contains three times x to be added, and the next contains seven times x to be added, making in all fifteen times x to be added; that is to say, $+15x$. In like manner, $-5x-3x-7x=-15x$; for, in this case, the $x$es are all to be subtracted. Also, $6ab+7ab-5ab-3ab+4ab=+9ab$; for $6ab$ with $7ab$ added will make, as above, $+13ab$;

and $5ab$ subtracted will leave $+8ab$; $3ab$ further subtracted will leave $+5ab$; and $4ab$ added to this will give $+9ab$. In like manner, $5ax^2 + 2ax^2 - 8ax^2 - 4ax^2 + 3ax^2 = -2ax^2$; and $10xy - 12xy + xy = -xy$. Since (Art. 7) it is indifferent in what order the terms are taken, we may first unite all the positive quantities into one term, and all the negative into another, and then subtract the latter from the former: thus, $5ax^2 + 2ax^2 - 8ax^2 - 4ax^2 + 3ax^2 = 10ax^2 - 12ax^2 = -2ax^2$.

24. The mark $>$ or $<$, placed between two quantities, denotes that the one to which the opening of the lines is turned is greater than the other. Hence $a > b$ means that a is greater than b; and $x < y$, that x is less than y.

25. When it is not known which of two quantities is the greater, the mark $\sim$ placed between them indicates that the less is to be subtracted from the greater: thus, $x \sim y$ indicates that if, of the two quantities, x be the greater, y is to be subtracted from x; but if y be the greater, x is to be subtracted from y; that is, the expression $x \sim y$ is equivalent to $x - y$ when $x > y$, and to $y - x$ when $x < y$.

PRAXIS.

26. The conversion of literal into numerical expressions will be found a useful exercise on the meaning of the symbols and operations of Algebra. The letters are supposed to denote particular numbers, which are substituted for the letters, and the same operations performed upon them that are performed upon the letters.

Thus let it be required to find what the expression

$$a + b^2 + ac - abx + \frac{abc}{5dx} - 2axy$$

will be in numbers, when

$$a = 4,\ b = 5,\ c = 3,\ d = 2,\ x = 1, \text{ and } y = 0.$$

Substituting the numbers for the letters, the expression becomes

$$4 + 5 \times 5 + 4 \times 3 - 4 \times 5 \times 1 + \frac{4 \times 5 \times 3}{5 \times 2 \times 1} - 2 \times 4 \times 1 \times 0\,;$$

or, performing the multiplications,

$$4 + 25 + 12 - 20 + \frac{60}{10} - 0\,;$$

by contracting, this is reduced to

$$41 - 20 + 6, \text{ or } 27,$$

which is the value of the expression on the above supposition respecting the letters.

EXERCISES.

1. What is $a + b - c$, when $a = 5$, $b = 4$, and $c = 6$?

2. What is the same expression, when $a = 7$, $b = 2$, $c = 8$; $a = 10$, $b = 6$, $c = 12$; $a = 9$, $b = 4$, $c = 13$; $a = 8$, $b = 3$, $c = 12$; $a = 1$, $b = 2$, $c = 6$?

3. What are $3a - 2c + b$; $4ab + 3ac - bc$; $3abc + 2ccc + 2aa + 3bb$, when a, b, and c, are 4, 5, 6; 3, 2, 4; 8, 6, 15; and 1, 9, 7?

4. If a, b, c, are 4, 5, 6; and 5, 7, 6, what is $a\,(c - b)$?

5. Find the difference between ab and $a + b$, and between abx and $a + b + x$, when $a = 7$, $b = 2$, $x = 9$; $a = 5$, $b = 8$, $x = 12$; $a = 1$, $b = 2$, $x = 3$; $a = \frac{1}{2}$, $b = \frac{1}{3}$, $x = \frac{1}{4}$.

6. Let $a = 6$, $b = 5$, $c = 4$, $x = 3$, $y = 2$, what are $4ax + 5cy - 7bx$; $5cxy - abc + 2aby - bcxy$; $2bbbc - 3aaxx + yyyyy$; $3abcxy + 2bcxy - 3acy - 6cxy$.

7. Let $a = 7$, $b = 4$, $x = 2$, what are $5abx + 7bx + 4ax - 5ab$; $3aabxx + 2bxxx - 5bbxx$; $2abx + 3bxx - 2aab + 11a - 5b$

8. Find $x^2 + 2xy + y^2$ in numbers; when $x = 4$, $y = 6$; $x = 5$, $y = 3$; $x = 10$, $y = 4$; $x = \frac{1}{2}$, $y = \frac{1}{3}$.

9. What is $x^2 - 2xy + y^2$, when $x = 4$, $y = 6$; $x = 9$, $y = 7$; $x = 13$, $y = 10$; and when $x = \frac{3}{4}$, $y = \frac{2}{3}$?

10. What is $n(n - 1)\,(n - 2)\,(n - 3)$, when $n = 4$; and when $n = 10$?

11. What is $s(s-a)(s-b)(s-c)$, when $s=\frac{1}{2}(a+b+c)$, and $a=4$, $b=7$, $c=9$; and again, when $a=12$, $b=8$, $c=6$?

12. Find the difference between a^2 and $2a$, when a is successively 8, 7, 6, 5, 4.

13. Find the difference between a^3 and $3a$, when a is 5, when a is 10, and when a is 12.

14. What is $abc+2c-3d+2ef$, when $a=1$, $b=2$, $c=3$, $d=4$, $e=5$, $f=0$?

15. What is $a^6+b^4-c^2-d+e+4abcdef$, on the same supposition?

16. What is $5ab+\frac{c+d-e}{2a}-2bf-2e$, on the same supposition?

17. What is $\sqrt{6bc}+\sqrt[3]{(bd)}+ab+3$, on the same supposition?

18. What is $\sqrt{(4e+b+c)}-\sqrt[3]{bc+2}-2d$, on the same supposition?

19. If $a=6$, $b=5$, $c=4$, $x=1$, $y=0$, what is $2a^2-5b-\frac{bc}{4x}-8a+y^2$?

20. What is $x^{10}+5y^4+\frac{abcx}{3cx}-\frac{abcxy}{5y^5}-2b$, on the same supposition?

21. What is $3\sqrt{c}+2a\sqrt{(2a+b-x)}$, on the same supposition?

22. What is $\frac{2b+c}{3a-c}-\frac{\sqrt{5b}+3\sqrt{c+x}}{2a+c}$, on the same supposition?

23. What is $[(a+b)x+(a-b)x]^2$, on the same supposition?

24. What is $n(n-1).(n-2).(n-3).(n-4)$, if n be 1, 2, 3, 4, 5, and 6 successively?

25. What must a and x be, so that $(a+x)^2$ may be 100?

26. What number must x be equal to, in order that abx may become ab?

27. Find the difference between $4x - 4$ and $4x \div 4$, when $x = 3$; and also when $x = 10$, and when $x = 16$.

28. What is $3ax + 5a^2$, if x be taken equal to a?

29. What is $3abx - 3b^3 + 4a^3 - 2a^2x$, when b and x become each equal to a?

30. What is $4xy - 8bd + 5x^2 + 2b^2 - 3y^2$, when x is put for y and b for d?

31. What does the expression $x + \frac{3x-5}{2} = 12 - \frac{2x-4}{3}$ become when $x = 5$?

32. What does $\sqrt{x} - \frac{8}{x} = \frac{7}{\sqrt{x} - 2}$ become when $x = 16$?

33. If $x = \frac{m+n}{m-n}$, what is x, when $m = \frac{1}{2}$, $n = \frac{2}{5}$?

34. If $y = \frac{3}{1+x} + \frac{3}{1-x}$, what is y, when $\frac{1}{2}$ is put for x?

35. Let $u = \frac{x}{y} - \sqrt{\frac{1+x}{1-y}}$, and $x = \frac{1}{4}$, $y = \frac{1}{5}$; what is u?

36. If $y = (\frac{3\sqrt{x}}{5} - 2)(\frac{1}{x-5})$, what is y, when $x = 25$; and when $x = 49$?

37. If $y = 5\sqrt{(62 + 3x)} - \frac{1}{2}\sqrt{(95\frac{2}{3} - 5x)}$, what is y, when $x = 6\frac{1}{3}$?

38. When $a = 3$, what is the difference between a^2 and $2a$; a^3 and $3a$; a^4 and $4a$?

39. Find the difference between a^5 and $5a$, when $a = 2$; and between a^2 and $2a$, if a be successively $\frac{1}{4}$, $\frac{1}{2}$, 1, 2.

40. What is $a^4 + 2ab + b^4$, when $a = 1$, $b = 2$; and also when $a = \frac{1}{2}$, $b = \frac{1}{4}$?

41. If $a = 6$, $b = 5$, $c = 4$, $x = 1$, $y = 0$, what are the expressions $\frac{2a+3c}{6x+4y} + \frac{4bc}{\sqrt{(2ac+c^2)}}$; and $\sqrt{(2a^2 - \sqrt{4c^2})} + bxy + \sqrt{(bc-c)} + x^5$?

42. Find the numerical value of each of the three following expressions, when $a=5$, $b=4$, $c=2$, $d=1$; $\frac{a^3}{b+d}-\frac{c^3}{d}+\frac{b^2+3c^2}{b+c+d}-2a+3d$; $\frac{a^2+b^2-d^2}{a+b+d}+\frac{abcd}{2b+c}-\frac{4a^2-10bc}{2c+d}$; $\frac{3bcd+2ab^2+16}{a+b-c+d}-\frac{a^3-b^3-c^3+7d^3}{4bc}$.

43. Find the difference between $6abc-2ab$ and $6abc\div 2ab$, when a, b, c, are 3, 5, and 6 respectively.

44. Find the difference between each pair of the following expressions: $(9+3)\times 8$ and $9+3\times 8$; $\sqrt{(49+15)}$ and $\sqrt{49}+15$; $(8+4)^2$ and $8+4^2$.

45. Find the difference between each pair of the following expressions if $a=6$ and $b=8$; $(a+b)^2$ and a^2+b^2; $a+b$ and $\sqrt{(a^2+b^2)}$; $(a+b)^3$ and $a+b^3$; $\sqrt{(a^2+b^2)}$ and $\sqrt{a^2}+\sqrt{b^2}$; $5(a+b)$ and $5a+b$.

46. Let s be as in exercise 11, what will $\sqrt{s(s-a)(s-b)(s-c)}$ be when $a=15$, $b=14$, $c=13$; and also when $a=50$, $b=40$, $c=30$?

47. If $x=6$, what is $1-\{1-(1-x)\}+2x-(3-5x)+2-(-4+5x)$?

48. When $x=4$, what is $\frac{1}{6}\{x(x+1)(x+2)+x(x-1)(x-2)\}+\frac{2}{3}(x-1)x(x+1)$?

49. Find the value of each of the following expressions, when $x=1$, $y=3$, $z=5$, $\sqrt[3]{(x+y+z)(z-y-x)\sqrt{(x+y+z)}}$; $\sqrt{[(x+y)^2+(z-y)^2+x]}$; and $\sqrt{\frac{4y^2-2z+2y}{2y+z-3x}}+\sqrt[3]{\frac{6y+2z-x}{3z-y^2+2x}}+\sqrt[4]{\frac{2z^2+3y^2+4x}{4z-\sqrt[3]{(3y-x)}-2x}}$.

50. Find what $\frac{5x}{9}-\frac{2x-1}{3}$ is, if $x=\frac{3}{8}$?

51. Find what $\frac{9x+7}{2}-(x-\frac{x-2}{7})$ is, if $x=9$.

52. If $x = 19$, what is $\frac{7x+5}{23} + \frac{9x-1}{10} - \frac{x-9}{5} + \frac{2x-3}{15}$?

53. What does the expression $\frac{6x+7}{15} - \frac{2x-2}{7x-6} = \frac{2x+1}{5}$ become, when 3 is substituted for x?

54. It will appear hereafter that whatever values a and b may have $\frac{a^2-b^2}{a-b} = \frac{a^3+b^3}{a^2+b^2-ab}$. Illustrate the truth of this by taking any numbers at pleasure for a and b.

Thus, let $a = 4$, $b = 2$, then each of the expressions becomes 6; and if $a = 5$, $b = 3$, each of them is 8; and the two will be always equal, whatever numbers are taken for a and b.

The following expressions will also be shown to be true hereafter; illustrate their truth in the same manner.

55. $(a+b)(a+b) = a^2 + 2ab + b^2$.

56. $(x+y)(x-y) = x^2 - y^2$.

57. $(n-m)(n-m) = n^2 - 2nm + m^2$.

58. $(a+b) + (a-b) = 2a$.

59. $(a+b) - (a-b) = 2b$.

60. $(a-4)(x+7) = x^2 + 3x - 28$.

61. A person is asked, "How much is it past 12 o'clock?" He looks at his watch, and observes that it wants 5 minutes of 12; how must he give a direct answer to the question?

62. How much is a merchant worth when his debts amount to £3000, and his entire property is only £2500?

63. If a quantity of goods cost £750, and be sold for £730, how much has the merchant gained?

64. How far is the town of Singapore south of the equator?

AXIOMS.

27. If equals be added to equals, the sums are equal; but if equals be added to unequals, the sums are unequal: thus, if $a = x$, then $a + b = x + b$, or if $a = x$ and $b = y$, then $a + b = x + y$; but if $a > x$, then $a + b > x + b$; and if $a < x$, then $a + b < x + b$.

28. If equals be subtracted from equals, the differences are equal; but if equals be subtracted from unequals, or unequals from equals, the differences are unequal: thus, if $a = x$, then $a - b = x - b$; or if $a = x$ and $b = y$, then $a - b = x - y$: but if $a > x$, then $a - b > x - b$; and if $a < x$, then $a - b < x - b$, &c.

29. From Art. 27, it follows, that if equals be multiplied by equals, the products are equal; but if equals be multiplied by unequals, the products are unequal: for multiplication is nothing else than a number of successive additions.

30. Hence, if equals be divided by equals, the quotients are equal: for by the nature of division, each dividend is a product, of which the divisor and the quotient are the factors; and since the divisors are equal, the remaining factors, the quotients, must be equal, else (Art. 29) the products, the dividends, could not be equal. For a similar reason, if equals be divided by unequals, or unequals by equals, the quotients are unequal.

31. Again, equal quantities raised to the same power are equal, and equal quantities raised to different powers are unequal: thus, if $a = x$, then $a^n = x^n$; but if $a > x$, then $a^n > x^n$; and if $a < x$ then $a^n < x^n$; for involution is only a succession

of multiplications. Also, the same roots of equal quantities are equal; for if it were otherwise, the given quantities, being obtained by raising those roots to the same power, could not be equal.

II. ADDITION.

32. *Connect the several quantities to be added by their respective signs, incorporating like quantities* (Art. 23).

First, Let the quantity to be added be simple. If it be required to add a to x; this, from the nature of algebraic notation, can only be done by placing the sign $+$ between them, which gives

$$x + a$$

for the sum. And if it be required to add $-a$ to x; this is done (Art. 6) by writing

$$x - a.$$

Secondly, Let the quantity to be added be compound. It is plain that to add a compound quantity is the same as to add its separate terms; that is,

$$x + (a + b) = x + a + b;$$

and $x + (a - b) = x + a + (-b)$; that is (Art. 6), $x + a - b$.

Thirdly, Let some of the quantities to be added be like. Thus, to $x + 3a$ add $y + 5a$: this would be done by writing $x + 3a + y + 5a$, which (Arts. 7 and 23) is the same as $x + y + 8a$.

It is often convenient to place the quantities to be added over one another: thus the last example may be written as under.

$$\begin{array}{r} x + 3a \\ y + 5a \\ \hline x + y + 8a \end{array}$$

EXERCISES.

1. $$\begin{array}{r} 2ax-3b^2y \\ 3ax-2b^2y \\ ax-b^2y \\ 5ax-4b^2y \\ ax-b^2y \\ \hline \end{array}$$

2. $$\begin{array}{r} 4a+2x \\ -3a-3x \\ a+7x \\ 2a-4x \\ -5a+x \\ 3a-6x \\ \hline \end{array}$$

3. $$\begin{array}{r} 3a+3x^2-b^3 \\ -5a-\ \ x^2+12b^3 \\ 7a+7x^2+13b^3 \\ -4a-4x^2-20b^3 \\ \hline \end{array}$$

4. $$\begin{array}{r} a+b-c \\ a-b+2c \\ -3a+2b-3c \\ 2a-b+2c \\ \hline \end{array}$$

5. $$\begin{array}{r} 4x^2y^2-5xy \\ 5ax+x^2y^2 \\ 7x^2y^2+3xy \\ -2ax-5x^2y^2 \\ \hline \end{array}$$

6. $$\begin{array}{r} 4a-2x+5b \\ -3x+2a-2y \\ 3b-5a+7x \\ -4a+2x-6b \\ \hline \end{array}$$

7. $$\begin{array}{r} -3a-4b+5c \\ 2b-4c+6ey \\ 7c-8d-9ey \\ 3b-\ \ a-3d \\ \hline \end{array}$$

8. $$\begin{array}{r} 4x(a+b)-1-3\sqrt{x} \\ 4\sqrt{x}+2x(a+b)-7 \\ 8-\sqrt{x}-5x(a+b) \\ 3x(a+b)-5\sqrt{x}-5 \\ \hline \end{array}$$

9. $$\begin{array}{r} ax+by+5x^2 \\ bx-3y+2x^2 \\ -3x^2-dx+dy \\ \hline \end{array}$$

10. $$\begin{array}{r} -12ax-5b \\ 3ax-5 \\ \hline \end{array}$$

11. $$\begin{array}{r} -8+a \\ 4 \\ \hline \end{array}$$

12. $$\begin{array}{r} 9-6x \\ 3x+x^2 \\ \hline \end{array}$$

13. $$\begin{array}{r} 15-8\sqrt{7} \\ 12\sqrt{7}-9 \\ \hline \end{array}$$

14. Add together $x + a - c$, $3b + 2d$, $-a + 4x$, $-3x + 5a$.

15. Add together $(a + b)\sqrt{x} - d\sqrt{(x^2 - y^2)} - mnp$, $2ax + 8mnp + 3(a + b)\sqrt{x}$, $p - 3d\sqrt{(x^2 - y^2)} + 2bx$, $-5mnp + 3p - 2(a + b)\sqrt{x}$.

16. Add together $3\sqrt{x} - \sqrt{(a + b)}$, $-2\sqrt{x} + \sqrt{(a + b)}$, $\sqrt{x} - 3\sqrt{(a + b)}$, $4\sqrt{x} + 2\sqrt{(a + b)}$.

17. Add together $3a\sqrt{x} - 2\sqrt[3]{y} - 5 + ax^2$, $ax + 11 - a\sqrt{x} + 3\sqrt[3]{y}$, $-2ax^2 - 4\sqrt[3]{y} + 4ax$, $-8 + 2a\sqrt{x} - 3ax + 2\sqrt[3]{y}$, $m\sqrt{x} - bx + 6 + 4ax^2$.

18. Add together a and b.
19. " " $a + b$ and $a - b$.
20. " " x and x^2.
21. " " n and $\sqrt{n}$.
22. " " $7a$ and $-8a$.
23. " " 1 and -2.
24. " " a and $-a$.
25. " " $3d$ and $4c$.
26. " " $6x$ and $-3y$.
27. " " $\sqrt{x}$ and $-2\sqrt[3]{x}$.
28. " " $-a^2$ and $-a$.
29. " " a^2x^3 and ax^3.
30. " " $\frac{1}{2}a - \frac{1}{2}b$ and $\frac{1}{2}a + \frac{1}{2}b$.
31. " " $a + b$ and $a - 2b$.
32. " " $x + p - y$ and $y - x$.
33. " " $6 + 7x$ and $-3x + x^2$.
34. " " $a^2 + ax$ and $x^2 - ax$.
35. " " $1 + \sqrt{x}$ and $1 - \sqrt{x}$.
36. " " $-11x^3$ and $-5x^2$.
37. " " $2a + b$ and $5a^2 - b$.
38. " " a^2 and a^3.
39. " " b^3 and $3b$.
40. " " $6a^n + 5\sqrt[n]{x} - \frac{4}{5}y$ and $-4\sqrt[n]{x} + \frac{1}{6}y - 2a^n$.

41. Add together $5a$ and $-11a$; also $-5a$ and $11a$.

42. „ „ $5axy$ and $7ax$.

43. „ „ -9 and 5.

44. „ „ $2x(a+b)$ and $3x(a+b)$.

45. „ „ $-19-x$ and x.

46. „ „ $x+a$ and $-a$; also $a-1$ and 1.

47. „ „ $-(a+x)y$ and $2(a+x)y$.

48. „ „ $5x+3b-a$ and $a-2b$.

49. „ „ $4b^4-1+2\sqrt{x}$ and $-3b^4+5-3\sqrt{x}$.

50. „ „ $3a^m-b^n+3x^p$, $2a^m-3b^n-x^p$, and $a^m+4b^n-x^p$.

51. „ „ $3a^2-5a+1$, $7a^2+2a-4$, and $13-a^2-4a$.

52. „ „ $a+b-c$, $a+c-b$, and $c+b-a$.

53. „ „ $6axy$ and $-4xay$.

54. „ „ $2a^n+1$ and $3a^{n+1}$.

55. „ „ $a+b+c-d$, $a+b+d-c$, $a+c+d-b$, and $b+c+d-a$.

III. SUBTRACTION.

33. *Change the signs of the subtrahend, and incorporate like quantities* (Art. 23).

First, Let the quantity to be subtracted be a simple quantity.

From x
subtract a.

This can only be done by writing $x-a$; that is to say,

$$x-(+a)=x-a.$$

Again, from x

subtract $-a$.

This (Art. 6) is the same as to add a, which is done by writing $x+a$; that is,

$$x-(-a)=x+a.$$

Secondly, Let the quantity to be subtracted be a compound quantity. It is plain that to subtract the compound quantity is the same as to subtract its terms in succession; that is,

$$x-(a+b)=x-a-b:$$

and

$x-(a-b)=x-a-(-b)$; that is, (Art. 6) $x-a+b$.*

EXAMPLES.

1. From $7a^2-4ax+11$ take $-3a^2+2ax+5$.

Here, changing the signs of the subtrahend, or conceiving them changed, and incorporating the like terms, we have $10a^2-6ax+6$, for the difference required.

$$\begin{array}{r} 7x^2-4ax+11 \\ -3a^2+2ax+\ \ 5 \\ \hline 10a^2-6ax+\ \ 6 \end{array}$$

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* The same may be shown otherwise, as follows:—

First, from x

subtract $a+b$.

If a be taken away, giving $x-a$, too little has been taken away by the quantity b; therefore b must further be taken away, which gives $x-a-b$.

Secondly, from x

subtract $a-b$.

If a be taken away, giving $x-a$, too much has been taken away by the quantity b; therefore the remainder $x-a$ is too small by b: hence b must be added, which gives $x-a+b$.

In the above general examples, the lower line, or subtrahend, contains only two terms, yet it exemplifies all the variety of signs that can occur in any compound quantity. The upper line, or minuend, is no less general; because, whatever number of terms it contained, they would be all set down unchanged in the difference, the change being made on the subtrahend only.

2. From $5a-6x$ subtract $2a-3x$. Here the required difference is $5a-6x-(2a-3x)=5a-6x-2a+3x=3a-3x=3\ (a-x)$.

EXERCISES.

1. $$\begin{array}{r} 12a-5x+7y-11 \\ \underline{-8a+4x-3y+12} \end{array}$$

2. $$\begin{array}{r} -4a+7b-6c+8\sqrt[4]{x} \\ \underline{-6a-3b-8c-5\sqrt[4]{x}} \end{array}$$

3. $$\begin{array}{r} 2a\sqrt{(x^2+y)}-5a^2x+11 \\ \underline{10-5a\ \sqrt{(x^2+y)}-8a^2x} \end{array}$$

4. $$\begin{array}{r} b-c+d+8 \\ \underline{19-b^2+c+d} \end{array}$$

5. $$\begin{array}{r} ax^2+bx+cxy-d \\ \underline{ex^2-fx+gxy-h} \end{array}$$

6. $$\begin{array}{r} -8a^2x-12\sqrt{b}+9 \\ \underline{-3a^2x-\ \ 4\sqrt{b}+5} \end{array}$$

7. $$\begin{array}{r} abx-5x^2+4\sqrt[3]{y} \\ \underline{ab\ \ -7x^2+2\sqrt{y}} \end{array}$$

8. $$\begin{array}{r} -6x \\ \underline{-11x-5} \end{array}$$

9.	From	$-9a$	take	$4a$
10.	„	$-9a$	„	$-4a$
11.	„	$9a$	„	$-4a$
12.	„	$9a$	„	$4a$
13.	„	$a+b$	„	$a-b$
14.	„	$a-b$	„	$a+b$
15.	„	b	„	c
16.	„	a	„	$-a$
17.	„	$-a$	„	$-a$
18.	„	x^2	„	x^2
19.	„	ab	„	a
20.	„	17		$5-a-3b$
21.	„	$4x$	„	4
22.	„	$3\sqrt{x}$	„	$2\sqrt{x}$
23.	„	a^2	„	a
24.	„	0	„	$-b$
25.	„	$3ab$	„	$3a$
26.	„	$2x$	„	x^2
27.	From	1	take	2
28.	„	$4a$	„	$6a$
29.	„	$4a$	„	$-6a$
30.	„	$-4a$	„	$-6a$
31.	„	$-4a$	„	$6a$
32.	„	5	„	6
33.	„	$\frac{3}{4}x$	„	$-\frac{1}{4}x$
34.	„	$-\frac{3}{4}x$	„	$\frac{1}{4}x$
35.	„	$-\frac{3}{4}x$	„	$-\frac{1}{4}x$
36.	„	$\frac{3}{4}x$	„	$\frac{1}{4}x$
37.	„	11	„	14
38.	„	$1+a$	„	$a+1$
39.	„	$1-a$	„	$a-1$
40.	„	0	„	$-(a-b)$
41.	„	7	„	-8
42.	„	$\frac{1}{2}a-\frac{1}{4}b$	„	$\frac{1}{2}b-\frac{1}{4}a$
43.	„	-11	„	6
44.	„	$\sqrt[6]{x}$	„	$\sqrt[5]{x}$

45. From $b^2 + 2ab + a^2$ take $b^2 - 2ab + a^2$
46. „ $\sqrt{x}\,(a-1)$ „ $a-1$

47. $1-(1-a)$
48. $1-(1+a)$
49. $1-(-1-a)$
50. $-1-(1-a)$
51. $1-a-(1+a)$
52. $-(1-a)-(1+a)$
53. $a-1-(-1-a)$
54. $1+a-(a-1)$

55. $\frac{1}{2}(a+b)-\frac{1}{2}(a-b)$*
56. $x-7-(2x-\overline{5-x})$
57. $a-(b-\overline{c+x})+(b-\overline{x-2b})$
58. $a-\{a-(a-\overline{a-x})\}$
59. $x^2+2xy+y^2-[x^2+xy-y^2-(2xy-x^2-y^2)]$
60. $2a\sqrt{(x-y)}-bxy-\overline{a\sqrt{(x-y)}-bxy}$
61. $4a-\sqrt{x}-(3a+2\sqrt{x})$
62. $a+b-[b+(a-b)]$
63. $4a^2-3b+7x-(6x+2b^2-4b+2a^2+a)$
64. $a+b-(2a-3b)-(5a+7b)-(-13a+2b)$
65. $1-\sqrt{x}-(\sqrt{x}+1)$
66. $1-[1-(1-\overline{1-x})]$
67. $6a-\{2a-[5a-(4a-a)]\}$
68. $3a-[a+b-\{a+b+c-(a+b+c+d)\}]$

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* From Ex. 55 and 13, and Exs. 35, 30, and 19 of Addition, in connection with Ex. 58 and 59 of the Praxis, it follows, that if the sum and difference of two unequal quantities be given, *the greater will be found by adding half the sum to half the difference, and the less, by taking half the difference from half the sum.* Thus, if 86 and 28 be respectively the sum and difference of two numbers, the numbers are 57 and 29; and if the half sum and half difference of two numbers be 32 and 14, the numbers are 46 and 18.

IV. MULTIPLICATION.

34. *If the signs of the factors are the same, the product is positive; and if the signs of the factors are different, the product is negative.*

First, Let it be required to multiply $+a$ by $+b$. This denotes that $+a$ is to be repeated as often as there are units in b; that is, that $+a$ is to be added b times. Now the sum of any number of positive quantities is positive (Art. 23); therefore the sum is $+ba$: Hence $(+a) \times (+b) = +ba$, or $+ab$ (Art. 11).

$$\begin{array}{l} +a \\ +a \\ +a \\ +a \\ +a \\ +\text{\&c. as far} \\ \text{as } b \text{ times.} \\ \hline +ba. \end{array}$$

Secondly, Let it be required to multiply $-a$ by $+b$. This denotes that $-a$ is to be added as often as there are units in b; and since the sum of any number of negative quantities is negative (Art. 23), this sum will be negative; that is to say, it will be $-ba$: hence $(-a) \times (+b) = -ba$.

$$\begin{array}{l} -a \\ -a \\ -a \\ -a \\ -a \\ -\text{\&c. to } b \\ \text{times.} \\ \hline -ba. \end{array}$$

Thirdly, Let it be required to multiply $+a$ by $-b$. This denotes that $+a$ is to be *repeated negatively** as often as there are units in b (the negative repetition being indicated by the sign of b); in other words, a is to be subtracted b times (Art. 6), or a is to be added b times, and the sum subtracted: hence $(+a) \times (-b) = -ba$.

$$\begin{array}{l} +a \\ +a \\ +a \\ +a \\ +a \\ +\text{\&c. to } b \\ \text{times.} \\ \hline -(+ba) = -ba \end{array}$$

---

* That is, added less than 0 times. This means that circumstances had been supposed to exist which would have required as many additions

Fourthly, Let it be required to multiply $-a$ by $-b$. This denotes that $-a$ is to be repeated negatively b times; that is, $-a$ is to be subtracted b times; or, in other words, $-a$ is to be added b times, and the sum subtracted: but (Art. 23) the sum is $-ba$: and this sum being subtracted will be $+ba$: hence $(-a)\times(-b)=+ba$.*

$$\begin{array}{l} -a \\ -a \\ -a \\ -a \\ -\text{\&c. to } b \\ \text{times.} \\ \hline -(-ba)=+ba. \end{array}$$

---

of a as there are units in b; whereas the real circumstances have turned out to be such as to require the same number of subtractions.

* This proposition may be demonstrated as follows after the method of multiplying compound quantities has been learned.

Let it be required to multiply $a-b$ by $c-d$.

Here $a-b$ is first to be multiplied by c: the product of $+a$ by $+c$ is $+ac$, as shown above; and the product of b by c is bc. But since a is greater than $a-b$ by b, the product ac is too great by bc; bc must therefore be subtracted: hence the product is $+ac-bc$.

Again, $a-b$ is to be multiplied by $-d$; but without regarding the sign of d, multiply $a-b$ by d, and the product is $ad-bd$, as before. Now, since c is greater than $c-d$ by d, in multiplying by c, a multiplier too great by d has been used: hence the product of $a-b$ by c is greater than the true product by the product of $a-b$ by d; the product of $a-b$ by d, or $ad-bd$, must therefore be taken from that of $a-b$ by c, or $ac-bc$: this gives

$$ac-bc-(+ad-bd)=ac-bc-ad+bd.$$

The following numerical illustration will make this proposition still more evident. Let it be required to multiply $14-8$ by $12-5$. This is obviously the same as to multiply 6 by 7; the product is therefore 42: but for the sake of illustration, let us proceed as in the foregoing general example. The product of 14 and 12 is 168, which is too great by 12 times 8, or 96: hence the true product is $168-96$ or 72, which is more simply formed; for $14-8=6$, and $6\times12=72$. Again, the product of $14-8$ by 5, is $70-40$, or $6\times5=30$. But since the first multiplier, 12, is too great by the second, 5, the first product, 72, is too great by the second, 30; 30 must therefore be subtracted, and $72-30=42$, the same as was at first obtained by following the simple and natural process.

$$\begin{array}{rl} & 14-8 \\ & 12-5 \\ \hline & 168-96=72 \\ \text{subtract} & 70-40=30 \\ \hline \multicolumn{2}{l}{168-96-70+40=72-30=42} \end{array}$$

35. *Powers of the same quantity are multiplied by adding their indices.*

For (Art. 14) $a^2 \times a^3 = aa \times aaa = aaaaa = a^5 = a^{3+2}$, and $a^7 \times a^4 = aaaaaaa \times aaaa = aaaaaaaaaaa = a^{11} = a^{7+4}$: and, in general, if a^n be multiplied by a^m, the product must contain as many factors, each equal to a, as are contained in a^n and a^m together ; that is, the number of such factors must be $n + m$, and therefore the product must be a^{n+m}

36. CASE I. When both multiplier and multiplicand are simple qualities,—*to the product of the numerical coefficients annex the product of the letters.*[*]

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[*] The following method of establishing the proposition that "the order of the factors does not affect the product," is given by Legendre, "Théorie des Nombres," p. 1 ; it is one of the demonstrations adverted to in the note to Art. 11.

Let there be a rectangular figure whose adjacent sides are divided into any number, a and b, of equal parts; then, if parallels be drawn from the points of division, the figure will be divided into rows of equal squares. Reckoning in one way, there will be a rows, each containing b squares, and the whole number will therefore be ab. But if we reckon in the other way, there will be b rows, each containing a squares, and the whole number will be ba. Now, the number of squares is always the same ; therefore, $ab = ba$.

Of this demonstration he observes, that it appears to him "à-la-fois claire, générale, et exacte ; cependant si on lui reproche d'être fondée sur des notions d'étendue étrangères à la science des nombres, voici comment on pourra y suppléer :" and he proceeds to give another demonstration founded on the assumption that $a \times 1 = 1 \times a$, to which, however, it may be objected that this is inadmissible as an axiom, and requires proof nearly as much as that $ab = ba$. Various other proofs have been given ; but they are equally unsatisfactory. No systematic proof, however, could render more obvious the truth of a proposition, so simple, and so nearly self evident.

EXAMPLES.

1. Multiply $5bc$ by $3a$.

Here (Art. 8) the product is $5bc \times 3a$, or (Art. 11) $5 \times 3 \times bc \times a$; that is, $15bca$, or (same Art.) $15abc$.

$$\begin{array}{r} 5bc \\ 3a \\ \hline 15\,abc \end{array}$$

2.
$$\begin{array}{r} -2xy \\ 6ab \\ \hline -12abxy \end{array}$$

3.
$$\begin{array}{r} 5a^2x^3 \\ -2a^3xy \\ \hline -10a^5x^4y \end{array}$$

4.
$$\begin{array}{r} -2\sqrt{x} \\ -3\sqrt{x} \\ \hline 6x \end{array}$$

5.
$$\begin{array}{r} a^2\sqrt{y} \\ a^3\sqrt{y} \\ \hline a^5y \end{array}$$

37. Case II. When the multiplier is simple, and the multiplicand compound,—*multiply the several terms of the compound quantity by the simple multiplier, and connect the products by their proper signs.*

This rule depends upon the principle stated in Art. 23.

EXAMPLES.

1.
$$\begin{array}{l} 2a^3 + 4a^2x^2 - 3b^2y \\ -3a^2x \\ \hline -6a^5x - 12a^4x^3 + 9a^2b^2xy. \end{array}$$

2.
$$\begin{array}{l} 2a + 3x\sqrt{y} - \sqrt{y} \\ 2\sqrt{y} \\ \hline 4a\sqrt{y} + 6xy - 2y. \end{array}$$

38. Case III. When both multiplier and multiplicand are compound quantities,—*multiply every term of the multiplicand by each term of the multiplier, and add the several products thus obtained.*

For (Arts. 11, 23) to multiply any quantity by the sum of any number of quantities, is the same as to multiply by those quantities separately, and then to add the products.

When different powers of the same quantity are given, the terms should be arranged according to the indices of those powers, in order that, in the partial products, like terms, when such occur, may be, as much as possible, under like terms.

EXAMPLES.

1. Multiply $a + b$ by $a + b$

	$a + b$
	$a + b$
Multiply $a + b$ by a	$a^2 + ab$
„ $a + b$ by b	$ab + b^2$
Add the partial products	$a^2 + 2ab + b^2$

2. Multiply $5x^4 - 4x^3 + 3x^2 - 3x + 1$ by $3x^2 - 2x - 1$.

	$5x^4 - 4x^3 + 3x^2 - 3x + 1$
	$3x^2 - 2x - 1$
Multiply each term by $3x^2$	$15x^6 - 12x^5 + 9x^4 - 9x^3 + 3x^2$
„ „ by $-2x$	$- 10x^5 + 8x^4 - 6x^3 + 6x^2 - 2x$
„ „ by -1	$- 5x^4 + 4x^3 - 3x^2 + 3x - 1$
Add the partial products	$15x^6 - 22x^5 + 12x^4 - 11x^3 + 6x^2 + x - 1$

3. Multiply $3a^5 + 2a^4 x - a^2x^3$ by $x^5 - ax^4 + 2a^3x^2$.

Here the terms being arranged according to the descending powers of a, and ascending powers of x, we have :—

$(3a^5 + 2a^4x - a^2x^2) \times (2a^3x^2 - ax^4 + x^5) = 6a^8x^2 + 4a^7x^3 - 3a^6x^4 - a^5x^5 + 2a^4x^6 + a^3x^7 - a^2x^8$.

4.
$$\begin{array}{r}
5-2\sqrt{3} \\
4+3\sqrt{3} \\
\hline
20-8\sqrt{3} \qquad \\
15\sqrt{3}-18 \\
\hline
2+7\sqrt{3}
\end{array}$$

5.
$$\begin{array}{l}
b+x \\
b-x \\
\hline
b^2+bx \\
\qquad -bx-x^2 \\
\hline
b^2-x^2
\end{array}$$

6.
$$\begin{array}{l}
a^n-2a^{n-2}x^2+3a^{n-4}x^4-4a^{n-6}x^6 \\
a^2-3x^2 \\
\hline
a^{n+2}-2a^nx^2+3a^{n-2}x^4-4a^{n-4}x^6 \\
\qquad -3a^nx^2+6a^{n-2}x^4-9a^{n-4}x^6+12a^{n-6}x^8 \\
\hline
a^{n+2}-5a^nx^2+9a^{n-2}x^4-13a^{n-4}x^6+12a^{n-6}x^8.
\end{array}$$

7. Multiply $4a^5c^2-3a^2bc^2+2c^3$ by $4a^5c^2+2a^2bc^2-c^3$

Ans. $16a^{10}c^4-4a^7bc^4+4a^5c^5-6a^4b^2c^4+a^2bc^5-2c^6$

8. What will $(a+b)x$ become when $a+c$ is put for x. Here $(a+b)x$ becomes $(a+b)\,(a+c)$, or $a^2+ab+ac+bc$.

9. What will x^2 become when x is changed into $x+h$. Here x^2 becomes $(x+h)^2$, or $(x+h)\,(x+h)$; that is $x^2+2xh+h^2$.*

---

* Since, by the nature of the process, the products of all binomial factors will be formed as in Exs. 1 and 9, we may conclude, in general, that *the square of a binomial is equal to the squares of its terms, and twice their product*. The same will hold when the terms are not both positive, for $(-a-b)^2=a^2+2ab+b^2=(a+b)^2$, and $(a-b)^2$ or $(b-a)^2=a^2-2ab+b^2$. The two forms may be combined in one expression by means of the double sign; thus: $(a\pm b)^2=a^2\pm 2ab+b^2$. By applying this formula we may often dispense with actual multiplication; thus we have directly, $(3a+2b)^2=9a^2+12ab+4b^2$; $(5-y^2)^2=25-10y^2+y^4$; $39^2=(30+9)^2=30^2+2\times 30.9+9^2=900+540+81$. We may square a trinomial in the same manner, by taking two terms as one; thus $(\overline{a+b}+c)^2=(a+b)^2+2(a+b)c+c^2=a^2+2ab$

39. The literal parts in the factors of a required product are often so related, that the law according to which the terms of the product will succeed one another is manifest, without actual multiplication. Thus in Ex. 2, last Art., the indices of x in each factor decrease by a unit from term to term, and the same takes place in the product: also in Exs. 3 and 6, the indices of a diminish, and those of x increase, from term to term in the factors and product. The same is shown by Ex. 1, while Ex. 7 does not present a like symmetry, either in the factors or product. A little consideration will show that in general the products will follow the law of the factors. In such cases, then, the literal part of the product will be known at once, and multiplication will be necessary merely to determine the coefficients. This is done by writing out in order the several coefficients, detached from the letters, with their respective signs, and operating upon them according to the Rule of Art. 38, the type, or form, of the process being precisely the same as if they were connected with the several letters. To the result so obtained the several powers are then subjoined. It is manifest, that if in adding the partial products, zero is obtained as the coefficient in any term, the continuity in the powers is interrupted, and that term will be wanting from the complete product. It is also plain, that if, in either factor, the powers do not succeed one another in regular order, one or more powers being wanting, zero must be put for the coefficient of such terms, in writing

$+ b^2 + 2ac + 2bc + c^2$:—and the same principle may be extended to quadrinomials.—*See Euc.* II., 4, and Exs. 55, 57, Art. 26.

Again, from Ex. 5 we see that *the product of the sum and difference of two quantities is equal to the difference of their squares;* and conversely, that *the difference of the squares of two quantities is equal to the product of the sum and difference of those quantities.* Thus for the product of $(4a + 3x) \times (4a - 3x)$ we have directly, without multiplication, $16a^2 - 9x^2$; and the factors of $25a^4 - 16b^6$ are $5a^2 + 4b^3$ and $5a^2 - 4b^3$. See Ex. 56, Art. 26.

out the detached coefficients. This being premised, multiplication by the method of detached coefficients will present no difficulty.

EXAMPLES.

1. Required the work of Ex. 2, last Art., by this method.

$$\begin{array}{rrrrrrr}
5 & -4 & +3 & -3 & +1 & & \\
3 & -2 & -1 & & & & \\
\hline
15 & -12 & +9 & -9 & +3 & & \\
 & -10 & +8 & -6 & +6 & -2 & \\
 & & -5 & +4 & -3 & +3 & -1 \\
\hline
15 & -22 & +12 & -11 & +6 & +1 & -1
\end{array}$$

Now the literal part of the first term being $x^4 \times x^2 = x^6$, and the powers forming a descending series, we have for the complete product,

$$15x^6 - 22x^5 + 12x^4 - 11x^3 + 6x^2 + x - 1.$$

2. Required the product as in Ex. 3, last Art., by this method.

In this Ex. the third term, namely a^3x^2, is wanting to complete the regular succession of powers in the first factor, and the second term, namely a^2x^3, in the second factor, and therefore in writing out the detached coefficients, zero must be put for the coefficients of these terms; the work will then be as on the margin. The double sign ± is prefixed to the coefficient 0, since it is doubtful what sign the wanting term

$$\begin{array}{rrrrrrr}
3 & +2 & \pm 0 & -1 & & & \\
2 & \pm 0 & -1 & +1 & & & \\
\hline
6 & +4 & \pm 0 & -2 & & & \\
 & \pm 0 & -3 & -2 & \pm 0 & +1 & \\
 & & & 3 & +2 & \pm 0 & -1 \\
\hline
6 & +4 & -3 & -1 & +2 & +1 & -1
\end{array}$$

would have had. Now the literal part of the first term being $a^5 \times a^3x^2 = a^8x^2$, and the terms being arranged according to the descending powers of a, and ascending powers of x, the rest of the literal part is easily supplied; the complete product then is

$$6a^8x^2 + 4a^7x^3 - 3a^6x^4 - a^5x^5 + 2a^4x^6 + a^3x^7 - a^2x^8.$$

3. Required the product, as in Ex. 6, by this method.

Here the indices of a in either factor diminish, while those of x increase, by 2 from term to term; and the same will hold in the product; the work therefore will proceed as on the margin. To the coefficients thus found the letters may then be annexed, as in the answer to Ex. 6. The question might also be worked by putting 0 for the coefficients of the intermediate terms, those, namely, in which the given powers would have the indices odd.

$$\begin{array}{rrrrr}
1 & -2 & +3 & -4 & \\
1 & -3 & & & \\
\hline
1 & -2 & +3 & -4 & \\
 & -3 & +6 & -9 & +12 \\
\hline
1 & -5 & +9 & -13 & +12
\end{array}$$

4. Find the product of $2a^3x - 2ax^3 - 3x^4$ and $3a^2 - 2ax + 3x^2$.

Here, in the first factor the term containing the second powers does not occur; and 0 is put for its coefficient. In the result 0 is found as the coefficient of the third and fifth terms; these therefore are wanting, and the literal part of the first term being $a^3x \times a^2 = a^5x$, the complete product will be

$$\begin{array}{rrrrrr}
2 & \pm 0 & -2 & -3 & & \\
3 & -2 & +3 & & & \\
\hline
6 & \pm 0 & -6 & -9 & & \\
 & -4 & \pm 0 & +4 & +6 & \\
 & & +6 & \pm 0 & -6 & -9 \\
\hline
6 & -4 & \pm 0 & -5 & \pm 0 & -9
\end{array}$$

$$6a^5x - 4a^4x^2 \qquad - 5a^2x^4 \qquad - 9x^6;$$

the spaces being left to indicate the terms that are wanting.

Many of the exercises that follow will afford practice on this method. It possesses considerable advantages over the common method when the terms contain two letters only; if there are more than two it may still be employed to some extent, but less conveniently.

EXERCISES.

Multiply

1. a by b
2. x by np
3. a^3 by a^5
4. a^5 by x^3
5. $7x$ by $8xy$
6. abc by def
7. $12d$ by $-4c$
8. $8x$ by $5x^5$
9. $2a^2$ by $5a^3$
10. a by -1
11. -11 by $-a$.
12. mn by pq
13. $4x$ by $\frac{1}{2}a$
14. $\frac{3}{4}b^2$ by $\frac{2}{3}ab^3$
15. a^n by a^p
16. $4b^{n-1}$ by $3b^{n+1}$
17. $-2b^3$ by $-4b^4$
18. b^{n-1} by b^2
19. x^{n+q} by x^{n-q}
20. a^{n-2} by a^{n+1}
21. x^n by x^3
22. a^{n+3} by a^{n-2}
23. a^nx^n by a^2x^2
24. $\sqrt{a}$ by itself
25. $3\sqrt{5}$ by $4\sqrt{5}$
26. $\frac{1}{2}a$ by $\frac{1}{2}a$
27. $2\sqrt{b}$ by $\sqrt{b}$
28. $4\sqrt{x}$ by $5\sqrt{x}$
29. $\frac{1}{3}a^2x^3$ by $\frac{1}{3}a^3x^5$
30. $4a$ by $\frac{1}{2}a$
31. $2\sqrt{ax}$ by $4ab$
32. $a^5x^2y^4$ by $a^{10}x^3y$
33. $4x-2b+3d^2$ by $2x$
34. $-7a^2+4abx-3d+8$ by $3a^3$
35. $a+bx+edy$ by ab
36. $7a-4ax+2by$ by $-4a^3x$
37. $4ax-2b^2y+7ad-5y$ by $3ab$
38. $4\sqrt{x}-3x^3y+5\sqrt[3]{x}$ by $2ay$
39. $2y^3-3xy+12ax-3$ by $\frac{1}{3}xy$
40. $y-3x+2az$ by my
41. $b^2y^3z-ay^3d+2y^5$ by $-bz$
42. $a+x$ by $a+x$
43. $a-x$ by $a-x$
44. $a+x$ by $a-x$
45. $6+a$ by $6-a$
46. $r-3$ by $2r-4$

47. $n - 4$ by $n + 4$
48. $4a - \frac{1}{2}$ by $4a - \frac{1}{2}$
49. $m + n$ by $m + p$
50. $a + b$ by $c + d$
51. $n - \frac{1}{4}$ by $n + \frac{1}{4}$
52. $\frac{1}{4}a - 1$ by $\frac{1}{4}a - 1$
53. $3b + 2y$ by $3b + 2y$
54. $5x^2 - 3y^3$ by $5x^2 + 3y^3$
55. $a^4 + b$ by $a^4 + b$
56. $x + 3$ by $x + 5$
57. $x + 4$ by $x - 11$
58. $x - 4$ by $x + 11$
59. $9 + 2\sqrt{10}$ by $9 - 2\sqrt{10}$
60. $a^n + b^n$ by $a - b$
61. $a^n + b^n$ by $a^n + b^n$
62. $1 - \sqrt{2}$ by $1 + \sqrt{2}$
63. $6a^2 + 10ax - 4x^2$ by $4a - 2x$.
64. $a^m + a^{m-1}x + a^{m-2}x^2 + a^{m-3}x^3$ by $a + x$.
65. $a^m - 2b^n$ by $2a^m - 3b^n$.
66. $x^2 + 4x + 8$ by $x^2 - 4x + 8$.
67. $1 + 2x + 3x^2 + 4x^3 + 5x^4$ by $1 - 2x + x^2$.
68. $a^4 - 2a^3b + 16b^4 - 8ab^3 + 4a^2b^2$ by $a + 2b$.
69. $5ab + 3ac - 4bc$ by $7ab - 18ac + 2bc$.
70. $x^5 - 5x^4 + 13x^3 - x^2 - x + 2$ by $x^2 - x - 2$.
71. $4x^3 + 3x^2 + 2x + 1$ by $x^3 - 2x^2 + 3x - 4$.
72. $(x^2 - 3x + 2)^2$ by $x^2 + 6x + 1$.
73. $a^5 - 5a^4 + 13a^3 - a^2 - a + 2$ by $a^2 - 2a - 2$.
74. $2a^3x^2 - 3a^4x^4 + 4a^5x^6 - 5a^6x^8$ by $ax^2 + 2a^2x^4 - 3a^3x^6$
75. $2x^3y^3 + 3xy^5 - 5x^4y^2$ by $7x^2y^3 - x^3y^2 + 2x^4y$.
76. $7a^3x - 3a^2x^2 + 2ax^3$ by $3a^2x^2 + ax^3 - x^4$.
77. $2x^3 + 4x^2 + 8x + 16$ by $3x - 6$.
78. $4x^3 - 29x + 4x^2 + 21$ by $2x - 3$.
79. $a^5x^3 + 2a^3x^2 + ax$ by $a^6x^3 - 4a^4x^2 + a^2x$.
80. $4v^4 + 5v^5 - 3v^6$ by $v - 3v^2 + 2v^3$.
81. $v^3 - 3v^2 + 4$ by $v^2 + 3v - 4$.
82. $5a^3 - 3ab^2 - 2b^3$ by $3a^4 + 2a^2b^2 - 5ab^3$.

Show that

83. $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$.
84. $(a^3 + a^2b + ab^2 + b^3)(a - b) = (a^2 + b^2)(a^2 - b^2)$.
85. $(a^2 + 3a + 1)^2 = a(a + 1)(a + 2)(a + 3) + 1$.
86. $(a + b + c)^3 - (a^3 + b^3 + c^3) = 3(a + b)(a + c)(b + c)$.

Multiply together continually

87. $x-1$, $x+2$, $x+3$, and $x-4$.
88. $x-5$, $x+6$, $x-7$, $x+8$.
89. $a+b+c$, $-a+b+c$, $a-b+c$, and $a+b-c$.
90. $2a-b$, $2a+b$, and $4a^2+b^2$.

V. DIVISION

40. *When the divisor and dividend have the same sign, the quotient is positive; when they have different signs, the quotient is negative.*

For, by the nature of division, the quotient must be such, that when multiplied by the divisor, it shall reproduce the dividend (Art. 34); therefore,

First, If $+ab$ be divided by $+a$, the quotient must be $+b$, in order that the product $+ab$ may be reproduced.

Secondly, If $-ab$ be divided by $+a$, the quotient must be $-b$, in order that the product $-ab$ may be reproduced.

Thirdly, If $+ab$ be divided by $-a$, the quotient must be $-b$, in order that the product $+ab$ may be reproduced.

Fourthly, If $-ab$ be divided by $-a$, the quotient must be $+b$, in order that the product $-ab$ may be reproduced.

41. *A power of a quantity is divided by any power of the same quantity, by subtracting the index of the divisor from the index of the dividend.*

For (Art. 14, 40) $a^5 \div a^3 = aaaaa \div aaa = aa = a^2 = a^{5-3}$; and $a^{11} \div a^7 = aaaaaaaaaaa \div aaaaaaa = aaaa = a^4 = a^{11-7}$; and, in general, if a^n be divided by a^m, the number of factors, each equal to a, in the quotient, will be found by subtracting the number of such factors in a^m from the number of such factors in a^n; that is, the number of factors will be $n-m$, therefore the quotient is a^{n-m}.

42. Case I. When both divisor and dividend are simple quantities; *take for the quotient such a quantity as, multiplied by the divisor, will reproduce the dividend.*

EXAMPLES.

1. Divide $20bxy$ by $5x$.

Here $20bxy$ being a product, and one of its factors $5x$, the other, by the nature of division, must be $4by$. The quotient, therefore, consists of the quotient of the numerical coefficients, with those letters annexed which are not common to the divisor and dividend.

$$5x)20bxy(4by$$
$$\underline{20bxy}$$
$$0$$

2. Divide $12a^4bn^3x^5$ by $-3a^2nx^4$.

$$-3a^2nx^4)12a^4bn^3x^5(-4a^2bn^2x.$$

3. Divide $-9b^3x^4$ by $-4ab^2x^3$.

Here -4 is contained $2\frac{1}{4}$ times in -9, and ab^2x^3 in b^3x^4 is contained $\frac{bx}{a}$ times, the common factors being expunged.

$$-4ab^2x^3)-9b^3x^4(2\tfrac{1}{4}\frac{bx}{a}, \text{ or } \frac{9bx}{4a}$$
$$\underline{-9b^3x^4}$$
$$0, \text{ or}$$
$$\frac{-9b^3x^4}{-4ab^2x^3}=\frac{9bx}{4a}$$

4. Divide $6bx^2y^3$ by $2abdx^2$.

Here $$\frac{6bx^2y^3}{2abdx^2}=\frac{3y^3}{ad}\times\frac{2bx^2}{2bx^2}=3\frac{y^3}{ad} \text{ or } \frac{3y^3}{ad}$$

5. Divide a by $\sqrt{a}$.

Here, as in other cases, the quotient must be such as to produce a when multiplied by $\sqrt{a}$; it must therefore be $\sqrt{a}$.

$$\begin{array}{r} \sqrt{a})a(\sqrt{a} \\ \underline{a} \\ 0 \end{array}$$

6. Divide x^n by x^{n-1}.

$$x^n \div x^{n-1} = x^{n-(n-1)} = x^{n-n+1} = x^1 = x.$$

43. CASE II. When the divisor is simple, and the dividend compound; *divide every term of the dividend by the divisor, and connect the quotients by their proper signs.*

EXAMPLES.

1. Divide $12ab^2c - 32bcx^2 - 4abcy - 4bc$ by $4bc$.

$$\begin{array}{l} 4bc)\underline{12ab^2c - 32bcx^2 - 4abcy - 4bc.} \\ \quad\quad 3ab \;-\; 8x^2 \;-\; ay \;-\; 1. \end{array}$$

2. Divide $-15n^4x^3 + 9n^3x^5 - 12n^2x^2 + 3n^2x^5$ by $-3n^2x^2$.

$$\begin{array}{l} -3n^2x^2)\underline{-15n^4x^3 + 9n^3x^5 - 12n^2x^2 + 3n^2x^5} \\ \quad\quad\quad\quad 5n^2x \;-\; 3nx^3 \;+\; 4 \;-\; x^3. \end{array}$$

44. CASE III. When both divisor and dividend are compound quantities; (1) *Place the quantities in one line, as in division in Arithmetic;* (2) *arrange, if possible, both divisor and dividend according to the powers of some letter common to both, so that its highest power may stand first, its next highest power second, and so on;* (3) *divide the first term of the dividend by the first term of the divisor; the quantity found is the first term of the quotient;* (4) *multiply this term into the divisor, and,* (5) *subtract the product from the dividend;* (6) *consider the remainder, if there be any, as a new dividend, and proceed as before;* and (7) *if a remainder do not disappear, indicate its division by the divisor, and annex the result to the quotient.*

EXAMPLES.

1. Divide $2ax + x^2 + a^2$ by $a + x$.

$$\begin{array}{l} a + x)a^2 + 2ax + x^2(a + x \\ \underline{a^2 + ax} \\ \phantom{a + x)a^2 + {}}ax + x^2 \\ \phantom{a + x)a^2 + {}}\underline{ax + x^2} \\ \phantom{a + x)a^2 + ax + {}}0 \end{array}$$

Arranging the terms according to the powers of a (Art. 38), and dividing a^2 by a, we find a for the first term of the quotient; (4) multiplying the divisor by this term a, we have $a^2 + ax$, which being subtracted (5) from the dividend, leaves the remainder $ax + x^2$ for the new dividend. Again, dividing (6) the first term ax of this remainder by the first term a of the divisor, we have x for the second term of the quotient, the sign being + because a and ax have the same sign; multiplying again, we have $ax + x^2$ to subtract, which leaves no remainder: the complete quotient is therefore $a + x$.

By this process, the dividend or given product, is analyzed into several partial products, and as the whole dividend is made up of these, the divisor is obviously contained in the whole as often as it is contained in all the parts; these products are subtracted from the dividend in succession, first, the product of the divisor $a + x$ by a, the first term of the quotient, and then the product of the divisor $a + x$ by x, the second term of the quotient; so that, in all, we take from the dividend the product of $a + x$ by $a + x$, without leaving any remainder: since therefore the quotient and divisor reproduce the dividend without remainder, this quotient is complete, and the operation is finished. If there be a remainder, the quotient is incomplete; and if the remainder do not disappear, on continuing the process, the quotient is interminable, and there is no series of terms which, multiplied by the divisor, can reproduce the dividend.* (*See* Art. 81).

---

* We might have arranged the terms according to the powers of x.

2. Divide $x^4 + a^4$ by $x + a$.

$$\begin{array}{ll}
x^4 + a^4 & (x + a \\
\underline{x^4 + ax^3} & x^3 - ax^2 + a^2x - a^3 + \dfrac{2a^4}{x+a} \\
\quad - ax^3 + a^4 & \\
\quad \underline{- ax^3 - a^2x^2} & \\
\qquad\quad a^2x^2 + a^4 & \\
\qquad\quad \underline{a^2x^2 + a^3x} & \\
\qquad\qquad\quad - a^3x + a^4 & \\
\qquad\qquad\quad \underline{- a^3x - a^4} & \\
\qquad\qquad\qquad\quad 2a^4 &
\end{array}$$

Here the divisor is put to the right of the dividend and the quotient below it. By this arrangement, long used in French works, room is saved, and the multiplication is more readily performed. The remainder $2a^4$ with the divisor subscribed is then annexed to the quotient. (*See* Art. 81.) In verifying the work by multiplying the divisor by the quotient, this remainder is added with the partial products.

The chance of error in this process, from neglecting to change the signs of the partial products before adding them, may be considerably lessened, by changing the sign of each term of the quotient, as it is set down; since in multiplying, the signs of the partial products will be thus at once

thus $x + a)x^2 + 2xa + a^2$,
and the operation would give $x + a$ for quotient.

If this mode of arranging the terms be disregarded, we shall sometimes find the complete quotient after a greater number of operations than are necessary: in most cases, however, the quotient will be interminable. The following arrangement of the terms of the last example will make this more plain:

$$\begin{array}{l}
a + x)2ax + a^2 + x^2(2x + a - x \text{ or } a + x \\
\qquad \underline{2ax + 2x^2} \\
\qquad \underline{a^2 - x^2} \\
\qquad \underline{a^2 + ax} \\
\qquad\quad - ax - x^2 \\
\qquad\quad \underline{- ax - x^2} \\
\qquad\qquad\quad 0
\end{array}$$

Here one operation more than was necessary is gone through, in consequence of the arrangement, and the quotient is reducible to $a + x$.

changed, and the subtractions converted into additions.* At the conclusion of the operation, the signs of the quotient must be again changed, in order to obtain the correct answer.

The work will then be as follows :—

$$\begin{array}{llll}
x^4 + a^4 & \multicolumn{3}{|l}{x + a} \\
\cline{2-4}
-x^4 - ax^3 & & -x^3 + ax^2 - a^2x + a^3 & \text{Quotient with contrary signs} \\
\cline{1-1}
\quad -ax^3 + a^4 & & x^3 - ax^2 + a^2x - a^3 & \text{Correct quotient} \\
\cline{3-4}
\quad +ax^3 + a^2x^2 & & x^4 + a^4 & \text{Dividend} \\
\cline{1-1}
\qquad a^2x^2 + a^4 & & -x^4 - ax^3 & \\
\qquad -a^2x^2 - a^3x & & +ax^3 + a^2x^2 & \text{Partial} \\
\cline{1-1}
\qquad\quad -a^3x + a^4 & & -a^2x^2 - a^3x & \text{products.} \\
\qquad\quad +a^3x + a^4 & & +a^3x + a^4 & \\
\cline{1-1}\cline{3-3}
\qquad\qquad 2a^4 & & \text{Sum } 2a^4 &
\end{array}$$

3. Divide $6a^3x - 17a^4x^2 + 30a^5x^3 - 25a^6x^4 + 12a^7x^5$ by $2a^2x - 3a^3x^2 + 4a^4x^3$.

$$\begin{array}{l|l}
6a^3x - 17a^4x^2 + 30a^5x^3 - 25a^6x^4 + 12a^7x^5 & 2a^2x - 3a^3x^2 + 4a^4x^3 \\
\cline{2-2}
6a^3x - 9a^4x^2 + 12a^5x^3 & 3a - 4a^2x + 3a^3x^2 \\
\cline{1-1}
\quad -8a^4x^2 + 18a^5x^3 - 25a^6x^4 & \\
\quad -8a^4x^2 + 12a^5x^3 - 16a^6x^4 & \\
\cline{1-1}
\qquad\qquad 6a^5x^3 - 9a^6x^4 + 12a^7x^5 & \\
\qquad\qquad 6a^5x^3 - 9a^6x^4 + 12a^7x^5 & \\
\cline{1-1}
\qquad\qquad\qquad\quad 0 &
\end{array}$$

Let this Ex. be also worked by the method explained in Ex. 2.

4. Divide $x^4 - 4x^3 - 34x^2 + 76x + 105$ by $x - 7$.

---

* The same object will be attained by changing the signs of the terms of the divisor. If this be done, it must be remembered, that in performing the successive divisions by the first term of the divisor, that term must be taken with its original sign, as otherwise the signs of the quotient would be reversed.

This question, as well as all others belonging to this third case, may be briefly worked as follows :—The first term only of the remainders (*sums* by the method of Ex. 2) being required in order to determine the successive terms of the quotient, we may, to find these, add the first terms mentally, and set down the partial products only, as on the margin in Ex. 2. The work will then stand as under :—

$$\begin{array}{lcl}
x^4 - 4x^3 - 34x^2 + 76x + 105(x - 7 & & \\
-x^4 + 7x^3 & & -x^3 - 3x^2 + 13x + 15. \\
\quad -3x^3 + 21x^2 & & \text{that is} \\
\qquad\quad +13x^2 - 91x & & x^3 + 3x^2 - 13x - 15. \\
\qquad\qquad\qquad +15x - 105 & & \\
\hline
\qquad\qquad\qquad\quad 0 & &
\end{array}$$

First terms of remainders, $3x^3, -13x^2, -15x$.

Here, the first term of the first remainder is $3x^3$; then $3x^3 \div x = 3x^2$; putting this in the quotient with its sign changed, and multiplying, we have the second partial product, $-3x^3 + 21x^2$. Adding now $21x^2$ to $-34x^2$, and dividing mentally, we have $13x$ for the next term of the quotient, and $13x^2 - 91x$ for the third product. Again, performing mentally the addition of the terms containing x, and then the division, and multiplying, we find $15x - 105$ for the fourth product. The sum of these partial products, and the given product, or dividend, is of course zero. The work is thus seen to occupy half the space taken up by the ordinary method. To aid the memory, the first terms of the sums (*remainders* by the uncontracted process) might be set down thus ($3x^3$, $-13x^2$, $-15x$), in the blank space to the right or left of the partial products, or in any other place, apart.

5. Divide $1 + ax + bx^2 + cx^3 + dx^4$, &c., by $1 - x$.

$$\begin{array}{l}
1-x)\,1+ax+bx^2+cx^3+dx^4,\text{ \&c.}\left(\begin{array}{l}1+(1+a)x+(1+a+b)x^2\\ +(1+a+b+c)x^3,\text{ \&c.}\end{array}\right.\\
\quad\underline{1-x\qquad\qquad}\\
\quad(1+a)x+bx^2\\
\quad\underline{(1+a)x-(1+a)x^2\qquad}\\
\qquad\qquad(1+a+b)x^2+cx^3\\
\qquad\qquad\underline{(1+a+b)x^2-(1+a+b)x^3}\\
\qquad\qquad\qquad\qquad(1+a+b+c)x^3+dx^4\\
\qquad\qquad\qquad\qquad\underline{(1+a+b+c)x^3-(1+a+b+c)x^4}\\
\qquad\qquad\qquad\qquad\qquad\qquad(1+a+b+c+d)x^4+ex^5,\text{ \&c.}
\end{array}$$

This quotient might be arranged as under:

$$\begin{array}{r|r|r|r|l}
1+1 & x+1 & x^2+1 & x^3+1 & x^4+,\text{ \&c.}\\
a & a & a & a & \\
 & b & b & b & \\
 & & c & c & \\
 & & & d &
\end{array}$$

45. When the given quantities are such, that the literal part of the terms of the quotient can be supplied by inspection (Art. 39), the method of "detached coefficients" may be advantageously employed in division. Thus, using the contracted mode of division given in Ex. 4,* the work of Ex. 3 will be as follows:—

$$\begin{array}{ll}
6-17+30-25+12\,\underline{|2-3+4\quad} & \\
-6+9-12 \qquad\qquad -3+4-3 & \text{quotient with contrary signs}\\
\quad+8-12+16 \qquad\quad 3-4+3 & \text{correct quotient}\\
\qquad\quad\underline{-6+9-12} & \\
\qquad\qquad\qquad 0 \qquad\qquad \text{Remainders, } -8,\ +6. &
\end{array}$$

---

* The ordinary uncontracted process may, however, be employed. Ex. 3 worked in this way will be as under;

$$\begin{array}{l}
6-17+30-25+12\,|\underline{2-3+4}\\
\underline{6-9+12}\qquad\qquad\ \ 3-4+3\\
\quad-8+18-25\\
\quad\underline{-8+12-16}\\
\qquad\qquad 6-9+12\\
\qquad\qquad\underline{6-9+12}\\
\qquad\qquad\quad 0
\end{array}$$

Here the process is conducted as already explained in Ex. 4; the literal part of the first term being a, and the order of the terms known, the literal part of the quotient is easily supplied. The first terms of the remainders, -8, and $+6$, are set down in a separate place.

1. Divide $6a^5x^3 - 19a^6x^5 + 48a^7x^7 - 44a^8x^9 + 33a^9x^{11} + 20a^{10}x^{13}$ by $2a^2x - 3a^3x^3 + 5a^4x^5$.

$$\begin{array}{l|l}
6 - 19 + 48 - 44 + 33 + 20 & 2 - 3 + 5 \\ \cline{2-2}
-6 + 9 - 15 & -3 + 5 - 9 - 4 \\
\quad + 10 - 15 + 25 & \;\;3 - 5 + 9 + 4 \\
\qquad - 18 + 27 - 45 & \\
\qquad\quad - 8 + 12 - 20 & \\ \cline{1-1}
\qquad\qquad 0 &
\end{array}$$

Hence, the literal part of the first term being a^3x^2, the complete quotient is $3a^3x^2 - 5a^4x^4 + 9a^5x^6 + 4a^6x^8$.

2. Divide $15a^9 + a^8b + 3a^6b^3 + 6a^5b^4 + 4a^4b^5 + 11a^3b^6 + 13a^2b^7 - 10ab^8 - b^9$ by $3a^3 + 2a^2b - 2b^3$.

Here, putting zero for the wanting term both in the dividend and divisor, the work will proceed as underneath.

$$\begin{array}{l|l}
15 + 1 \pm 0 + 3 + 6 + 4 + 11 + 13 - 10 - 1 & 3 + 2 \pm 0 - 2 \\ \cline{2-2}
-15 - 10 \pm 0 + 10 & -5 + 3 - 2 - 3 \\
\quad + 9 + 6 \pm 0 - 6 & \;\; + 2 - 4 - 3 \\
\qquad - 6 - 4 \pm 0 + 4 & \qquad \text{or} \\
\qquad\quad - 9 - 6 \pm 0 + 6 & \;5 - 3 + 2 + 3 \\
\qquad\qquad + 6 + 4 \pm 0 - 4 & \;\; - 2 + 4 + 3 \\
\qquad\qquad\quad - 12 - 8 \pm 0 + 8 & \\
\qquad\qquad\qquad - 9 - 6 \pm 0 + 6 & \\ \cline{1-1}
\pm 0 \pm 0 \pm 0 \pm 0 \pm 0 \pm 0 \pm 0 + 3 - 2 + 5 &
\end{array}$$

Now, as the literal part of the first term of the quotient is a^6, there will be in all seven terms, six containing the powers of a, and one independent of a; the three last coefficients are therefore those of a remainder. In fact, the process must terminate at the term whose coefficient is 3, as it contains a^2, a power of a lower than that in the first term of the divisor. The complete quotient is therefore,

$$5a^6 - 3a^5b + 2a^4b^2 + 3a^3b^3 - 2a^2b^4 + 4ab^5 + 3b^6,$$

and the remainder, $3a^2b^7 - 2ab^8 + 5b^9$.

EXERCISES.

Divide

1. bc by c.
2. x^3 by x.
3. $-x$ by itself
4. $20x^3y$ by $5xy$
5. $-x$ by x.
6. $15ax$ by $3x$
7. $12b^5$ by $4b^4$
8. $15a^3b^2c$ by $5a^2b$
9. $\sqrt{x}$ by $\sqrt{x}$
10. a^6 by a^5
11. $8x$ by $2x$.
12. b^{2n} by b^n
13. a^n by a
14. a^3n by $-a^3n$
15. $24a^2b^3$ by $8a^2b^2$
16. $12abcdef$ by $4acf$
17. x^a by x^b
18. $72ab^5x^3yz$ by $9b^5x^2z$
19. $-a$ by -1
20. $-30a^nx^my^5$ by $-6x^my^5$
21. -1 by -1
22. 1 by -1
23. $5a^{n+1}$ by $5a^{n-1}$
24. 2 by $\sqrt{2}$
25. a^{n-1} by a^{n+1}
26. $12a^nx^5y^2$ by $4a^2x^3y$
27. $8a$ by $\sqrt{8a}$
28. twice the cube of a by half its square
29. $-15b^5x^3y$ by $3b^2x^3$
30. $12ax^3$ by $4bx$
31. $4b$ by $3b$
32. x by $4x$
33. nm by np
34. x^n by x^{n-1}
35. x^{n-1} by x^{2n}
36. abc by ad.
37. $8x$ by $4\sqrt{x}$

Divide

38. b^4 by b^7
39. $-16a$ by $-8b$
40. $12a$ by $-4b$
41. a^9 by a^2
42. a^7 by x^4
43. $3b\sqrt{y}$ by $2a\sqrt{y}$
44. a^3 by a^5
45. $b^3xy^3p^2q^3$ by $5p^2x$
46. $8mpx$ by $2mxy$
47. $36a^{10}b^9x^6y^2$ by $9a^3b^5x^3y$
48. $-3a^2b$ by $-3a^2b$
49. -1 by a
50. $-a$ by $-b$
51. $12b^2y+4by^2-16b$ by $4b$
52. $3abc + 12abx - 9a^2b$ by $3ab$
53. $-9ab^3x^4y + 12b^2x^3y^2 + 15a^3b^4x^5y - 3b^2a^2x^4y^3$ by $-3b^2x^3y$.
54. $ab+b^2$ by $2b$.
55. $a^2-2a+7-3x-1$ by -1.
56. $by^n + by^{n+1} + by^{n+2} + by^{n+3} +$ &c. by y^n.
57. $-3a+7x-10y^2+2z-1$ by -1.
58. $7n^2x^3 - 8xy^3 - 12bxy^2 + 13mnx$ by $4nx$.
59. $5a^2x^3y^4 - 10x^5y^4 - 20a^5x^ny^3$ by $5x^3y^3$.*
60. $a^2-2ab+b^2$ by $a-b$.
61. $x^2+3x-28$ by $x-4$.
62. $x^2+2xy+y^2$ by $x+y$.
63. m^2-36 by $m+6$.
64. $n^2+2np+p^2$ by $n+p$.
65. $a-1$ by $\sqrt{a}+1$.
66. $x^3-9x^2+27x-27$ by $x-3$.
67. $a^{n+1}+ab^n-a^nb-b^{n+1}$ by a^n+b^n.
68. $a^{n+1}+2a^nx+2a^{n-1}x^2+2a^{n-2}x^3$, &c. by $a+x$.

---

* Such examples as the following come under the Second Case, though the divisor has a compound form; for every term is divisible by the compound factor:

$$1.\quad 3b(a+x)\overline{)3ab(a+x)^3 - 6b(a+x)^2 + 3b(a+x)}$$
$$a(a+x)^2 - 2(a+x) + 1$$

$$2.\quad a+b\overline{)6a(a+b)^3 - 8(a+b)^2 - 3x(a+b)}$$
$$6a(a+b)^2 - 8(a+b) - 3x$$

69. $1-5x+10x^2-10x^3+5x^4-x^5$ by $1-2x+x^2$.

70. $4x^3+4x^2-29x+21$ by $2x-3$.

71. $1-n^2$ by $1-n$.

72. x^4-a^4 by $x-a$.

73. x^4-a^4 by $x+a$.

74. $x^{pq}-1$ by x^q-1.

75. x^4+a^4 by $x-a$.

76. $6x^4-96$ by $3x-6$.

77. a^6-x^6 by $a-x$.

78. n^5-p^5 by $n-p$.

79. n^5+p^5 by $n+p$.

80. x^9+a^9 by $x+a$.

81. x^6-1 by $x-1$.

82. a^n+b^n by $a+b$.

83. $12a^2+26ab-36ac+18ad-10b^2+29bc-6bd-21c^2+9cd$ by $6a-2b+3c$.

84. $8a^5x^5+6a^6x^4-15a^7x^3+a^9x$ by $4a^2x^3-3a^3x^2-a^4x$.

85. $6x^5-x^4y-13x^3y^2+10x^2y^3-2xy^4$ by $2x^3-3x^2y+xy^2$.

86. $4ax^3+51a^2x^2+10a^4-48a^3x-15x^4$ by $4ax-5a^2+3x^2$.

87. $21x^3y^2+25x^2y^3+68xy^4-40y^5-56x^5-18x^4y$ by $5y^2-8x^2-6xy$.

88. $x-5x^3+7x^5-5x^7-6x^9+28x^{11}$ by $1-2x^2-4x^4$.

89. $12a^5b-b^6+2ab^5+8a^4b^2-9a^2b^4-13a^3b^3$ by $3a^2+2ab-b^2$.

90. $a^9-4a^7x^2+5a^6x^3-5x^9+14ax^8-20a^2x^7-10a^4x^5+19a^3x^6$ by $a^4+5x^4+2a^3x-4ax^3-3a^2x^2$.*

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* From a comparison of Exercises, 72 to 82, and Example 2, it will appear that the difference of two equal powers is always divisible by the difference of the roots; that the difference of two equal powers is divisible by the sum of the roots, when the index is an even number; and that the sum of two equal powers is divisible by the sum of the roots, when the index is odd; but that in other cases there will be a remainder, after the division. That is,

$x^n - a^n$ is *always* divisible by $x - a$.
$x^n - a^n$ is divisible by $x + a$ if n be an *even* number
$x^n + a^n$ is divisible by $x + a$ if n be an odd number.

A general proof of these statements will be given hereafter.

MISCELLANEOUS EXERCISES.

1. What is the sum of the sum and difference of two quantities?

2. What is the difference of the sum and difference of two quantities?

3. If the less of two quantities is x, and the difference 12, what is the greater?

4. If y be the greater of two quantities, and d the difference, what is the less?

5. What is the difference between the square of the sum, and the sum of the squares, of two quantities?

6. Find the difference between the square of the sum, and the square of the difference, of two quantities?

7. Find the sum of the square of the sum, and the square of the difference, of two quantities?

8. If p be contained q times in a with a remainder r, what is the expression for a?

9. What does $p^2 + q^2$ want of being a complete square?

10. What must be added to $x^2 + 12x$ to make a complete square?

11. x persons have £a to pay, what is the share of each?

12. Express the two numbers whose digits are x and y; and x, y, and z.

13. What are the factors of $a^2 - x^2$, of $64 - x^2$, of $4n^2 - 64$, of $x - 100$, and of $81a^4 - 100x^2$.

14. Resolve $x^3 - y^3$ into two factors; $x^4 - y^4$ into three, $a^6 - x^6$ and $a^8 - x^8$ into four, and $a^{16} - x^{16}$ into five factors.

15. A person paid a shillings for x articles, what is the price of each?

16. Find the squares of $6a - 4x$, $2b - 5y^3$ and $\frac{1}{2}a - 8$, without actual multiplication.

17. What does $9 - 6x$ want of being a complete square?

18. If $s = \frac{1}{2}(a + b + c)$, what will $2s - 2a$ be; and also $2s - 2b$, and $2s - 2c$?

19. Shew that the half sum of two quantities exceeds the less as much as the greater exceeds the half sum.

20. If a man can do a piece of work in 12 days, how much of it will he do in 5, in 7, and in x days?

21. What are the factors of $a^2 - (b - c)^2$, and of $(b + c)^2 - a^2$?

22. Resolve $4a^2b^2 - (a^2 + b^2 - c^2)^2$ into four factors.

23. The sum of two numbers is 356 and their difference 134, what are the numbers?

24. The half sum and half difference of two numbers are 218 and 124; find the numbers.

25. The down-train from London, and the up-train from Edinburgh, start at the same hour, and meet at the end of x hours; the former travelling 42 miles per hour, and the latter 38; express the whole distance in terms of x.

26. At an election 3428 persons voted, and the successful candidate had a majority of 312 votes; how many voted for each.

27. If each of three quantities be subtracted from half their sum, what will be the sum of the three remainders?

28. Find the same in the case of four, and in that of five quantities.

29. What will x^3 become when $x + h$ is put for x?

30. What does $(a + x)(a - x)$ become when $a + b$ is substituted for x?

31. If $a - b$ be substituted for x in the expression $ax - bx$, what will that expression become?

32. If $x + h$ be substituted for x, in each of the expressions x^4, x^5, x^6, what will be the first terms of the difference between x^4 and $(x + h)^4$, x^5 and $(x + h)^5$, x^6 and $(x + h)^6$?

VI. MEASURES AND MULTIPLES.

ART. 46. When a less quantity is contained a certain number of times exactly in a greater, the greater is called a *multiple* of the less, and the less is said to be a *part*, *measure*, or *sub-multiple* of the greater. Thus, b is a measure or sub-multiple of ab, and ab a multiple of a, and also of b : also, 36 is a multiple of 9, and 9 a part, measure, or sub-multiple of 36 : but 9 is not a measure of 40.

47. When one quantity divides several others, it is called their *common* measure, or divisor; and the *highest* common divisor of several quantities, is the factor of *highest dimensions* (Art. 17) common to them. Thus, 9 is a common measure or divisor of 54 and 72; and a^2-x^2 the highest common divisor of a^4-x^4, and $a^3+a^2x-ax^2-x^3$. In conformity with the usage of arithmetic this highest divisor is often called the *greatest* common measure.

48. When a quantity is a multiple of several others, it is called their *common multiple*: thus, 56 is a common multiple of 4, 7, 8, 14, and 28; also $12a^2b^2xy$ is a common multiple of $6a^2x$, $4b$, $3ax$, and $2by$.

49. An *even* number is one which can be divided into two equal integral parts; and an *odd* number, one which cannot be so divided.

50. Hence, if n represent any number whatever, even or odd, $2n$ will be always even, as it is divisible by 2. Hence, also, a number greater or less than $2n$ by unity, will be odd: thus, if $n=7$, $2n+1=15$, and $2n-1=13$, both odd; if $n=18$, $2n\pm1=37$, or 35, both odd, &c.; so that all odd numbers may be represented by the expression $2n\pm1$.

This is generally expressed by saying that even numbers are of *the form* $2n$, and odd numbers of *the form* $2n \pm 1$. Even and odd numbers may, however, be represented by other expressions besides these; for, if n and n' be any two numbers, $2(n \pm n')$ will be even; but $2(n \pm n') \pm 1$ will be odd. Thus, if $n = 9$, and $n' = 6$, $2(n \pm n') = 30$ or 6, even, and $2(n \pm n') \pm 1 = 31$ or 29, 7 or 5, all odd. So also $2(2nn')$, $2(n \pm n' \pm 1)$, $2(nn' \pm n \pm n')$, are even forms, or of the form $2n$; and $2(n \pm n' \pm 1) \pm 1$, $2(2nn' \pm n) \pm 1$, &c., are odd forms, or of the form $2n \pm 1$.

51. A *prime* number is one which is not composed of any integral factors; and a *composite* number, one which is composed of two or more such factors.

52. Hence, a prime number cannot be divided by any numbers but itself and unity. Hence, also, no even number but 2 is prime.*

53. *Commensurable* quantities are those which have a common measure greater than unity; and *incommensurable* quantities, or quantities *prime to each other*, are those which have no common measure greater than unity. Thus, 6 and 8 $4ax$ and $5ax^2$, are commensurable; while 5 and 6, 7 and 2, ax and by, $\sqrt{3}$ and $\sqrt{8}$, 2 and $\sqrt{5}$, 1 and $\sqrt{2}$, are incommensurable, or prime to one another.

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* Those factors of composite numbers which are themselves composite, can always be resolved into the same prime numbers. Thus, the factors of 30 are 5 and 6, and 3 and 10; now $6 = 3 \times 2$, therefore $30 = 5 \times 3 \times 2$, the factors being all prime. Also, $10 = 5 \times 2$, therefore $30 = 3 \times 5 \times 2$, the same prime factors as before. Thus, also, $360 = 2 \times 180$, $180 = 2 \times 90$, $90 = 2 \times 45$, $45 = 3 \times 15$, $15 = 3 \times 5$; therefore $360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5$; and whatever composite factors of 360 are taken, they can be always resolved into these same prime numbers.

54. *If a quantity measure another, it will measure any multiple of it.*

For if $a = bc$, then $ma = mbc$; or if b be contained c times in a, it will be contained mc times in ma.

55. *If a quantity measure each of two others, it will also measure their sum and difference, or the sum and difference of any multiples of them.*

Let p divide a and b, and let the quotients be q and q' then $\frac{a}{p} = q$, $\frac{b}{p} = q'$; therefore, by the nature of division, $a = pq$ and $b = pq'$ whence (Art. 27, 28) $a \pm b = pq \pm pq'$, or $a \pm b = p(q \pm q')$; or (Art. 30), dividing the equals by p, $\frac{a \pm b}{p} = q \pm q'$. Now, since by hypothesis a and b are divisible by p, and q and q' are the quotients, q and q' are whole numbers; therefore their sum and difference are also whole numbers; that is, $q \pm q'$ is an integer, and consequently $a \pm b$ is divisible by p.

Again, let ma and nb represent any multiples of a and b; then, since $a = pq$, and $b = pq'$, $ma = mpq$, and $nb = npq'$: hence

$$ma \pm nb = (mq \pm nq')p,$$

and

$$\frac{ma \pm nb}{p} = mq \pm nq';$$

but m, n, q, and q', being integers, the quotient $mq \pm nq'$ is also an integer, and $ma \pm nb$ is therefore divisible by p.*

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* Thus, if a and b be 54 and 42, then 6, which is a divisor of both, divides also 54 ± 42, or 96 and 12, the quotients being 16 and 2, both integers. And if m and n be 4 and 5, we have $\frac{216 \pm 210}{6} = 36 \pm 35$, or $\frac{426}{6} = 71$, and $\frac{6}{6} = 1$, integers as before.

56. It follows, that if a number divide the whole of another number and a part of it, it will also divide the other part; and that if a number consist of several parts, each of which has a common divisor, p, then will the whole number composed of these parts be divisible by p.

57. *The highest common divisor of two quantities,* A *and* B, *is the highest common divisor of* Aa *and* Bb, *a and b being integer multipliers, prime to one another, and such that there is no divisor common to a and* B, *or to b and* A.

For if d be the highest common divisor of A and B, so that $\frac{A}{d} = p$ and $\frac{B}{d} = q$, or $A = pd$ and $B = qd$, p and q can have no common measure. We have similarly $\frac{Aa}{d} = ap$ and $\frac{Bb}{d} = bq$ or $Aa = apd$ and $Bb = bqd$, and ap and bq can have no common measure; no new factor is introduced, and those common to A and B are common also to Aa and Bb, and the same is obviously true of A and Bb, or Aa and B, for this is only supposing either of the multipliers to be unity. Again, conversely, on the same supposition, the G. C. M. of Aa and Bb is also the G. C. M. of A and B. For if $\frac{Aa}{d} = ap$ and $\frac{Bb}{d} = bq$, then $\frac{A}{d} = p$ and $\frac{B}{d} = q$, and since ap and bq have no common measure, it follows, that p and q also have no common measure; and the G. C. M. of Aa and Bb is also the G. C. M. of A and B.*

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* Thus, if we take the numbers 30 and 36, whose G. C. M. is 6; and if we multiply them respectively by 7 and 11, the products $6 \times 5 \times 7$ and $6 \times 6 \times 11$, or 210 and 396, have the same common measure, 6, as before, the multipliers being prime to one another, and to the given

58. To find the highest common divisor of two algebraic quantities. FIRST, *If the divisor sought be* SIMPLE, *to the greatest common measure of the numerical coefficients annex the letters common to all the quantities.*

Thus, the greatest common measure of $24ab^4n^2y^5$ and $56an^2x^2y^3$ is $8an^2y^3$.

Secondly, If the divisor sought be *compound,* the following rule must be employed :—

(1.) *Divide either quantity, or both, by the highest simple factor contained in it* (Art. 57); (2) *and having arranged the terms as in Division* (Art. 44), *divide the one of higher by the one of lower dimensions, if their degrees be different*; (3) *if there be a remainder, divide the last divisor by it*; (4) *continue the process till there is no remainder, dividing always the preceding divisor by the preceding remainder, and at the same time multiplying or dividing the dividend, and the successive divisors and dividends by any quantity which has no factor in common with the original divisor, or dividend* (Art. 57); *the last divisor will be the common divisor required.*

The following is the general form of the process: the multipliers, a, b, c, . . . fulfilling the conditions of Art. 57, and thus rendering the quotients, p, q, . . . integral in every

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numbers. Also, if we multiply 30 by 5, and 36 by 9 the products $6 \times 5 \times 5$ and $6 \times 6 \times 9$, have still 6 as their G. C. M.; whereas if we multiply 30 by 9 and 36 by 5, the products $6 \times 5 \times 9$ and $6 \times 6 \times 5$, or 270 and 180, have $6 \times 5 \times 3$ or 90 for their G. C. M., a factor of each number being introduced into the multiple of the other, which is contrary to the hypothesis of a being prime to B and b to A. Further, if one only of the given numbers, as 30, be multiplied by any number, as 5, which has no factor in common with 36, the G. C. M. of 30, and the product 150, is not altered.

case, fractional results being excluded by the nature of the operation. (Art. 54, 55, 57.)

Here, B being supposed to represent the quantity of lower dimensions, and a a factor introduced to avoid fractions, B is contained p times in Aa with a remainder Cc; where C is a compound quantity, and c a quantity, either simple or compound, and not a factor of both A and B; take C alone for the new divisor, and for the new dividend the last divisor B or a multiple, Bb, of it, if necessary; and so on till there is no remainder. Then the last divisor, D, is the highest common divisor of A and B.[*]

$$\begin{array}{l} B)A \\ B)Aa(p \\ \quad \underline{pB} \\ \quad Cc)B \\ \quad\ C)Bb(q \\ \qquad \underline{qC} \\ \qquad Dd)C \\ \qquad\ D)C(r \\ \qquad\quad \underline{rD} \\ \qquad\quad \cdot\ \cdot \end{array}$$

To prove this it is necessary to shew that—

1. Every measure of A and B is a measure of D. For whatever quantity, m, measures A and B, it will measure Aa and pB (Art. 54), and therefore, $Aa-pB$ (Art. 55), that is Cc; and since m measures Cc it measures also C, because, by hypothesis, c is prime to A and B, and therefore also to m, which is a measure of A and B. Now m, measuring B and C, will measure also Bb and qC, and therefore also their difference Dd; and since m measures Dd it will measure D, for d is prime to A and B and

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[*] That the above process must at last terminate, may be proved as follows:—Rejecting the multipliers a and c for the sake of simplicity, we shall have $A=pB+C$. Now, $B>C$ and therefore $pB>C$, hence $pB+C>2C$, that is, $A>2C$, and $\frac{1}{2}A>C$, or $C<\frac{1}{2}A$; therefore $pB>\frac{1}{2}A$, and from A a quantity greater than its half has been taken. In the same manner it might be shewn, that when C is made dividend more than its half is taken away, and so on with respect to every successive dividend, so that at last there will remain a quantity less than anything that can be assigned.

therefore to m, their measure. Thus, any measure, m, of A and B is a measure also of D.

2. D is a measure of A and B.

For D is by hypothesis a measure of C, being contained in it without remainder; it is evidently a measure of Dd; it therefore measures $qC + Dd$ or Bb, and consequently B also, since d is prime to A and B, and therefore to D, the measures of A and B having been just proved to be those of D. Now D measuring pB and Cc measures Aa, their sum, and therefore A also, since a is prime to A and B, and therefore to D. But it has been already shewn that D measures B; it therefore measures both A and B.

3. D is the greatest common measure of A and B.

For since every common measure of A and B is a measure of D, and since no quantity greater than D can measure it, therefore every common measure of A and B is *not greater* than D; now D is a common measure of A and B, and therefore there is none greater than it; that is, D is the greatest common measure of A and B.*

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* The arithmetical process for finding the greatest common measure of two numbers is of the same form as that in the text, but much less complicated; a proof is subjoined.

Let a and b be the numbers whose common measure is required; and let a be greater than b; the operation will be conducted as follows:—

Dividing a by b, let the quotient be p, with a remainder c; again, making this remainder the divisor, and the former divisor the dividend, let c be contained in b, q times, with a remainder, d; and repeating the process, let d be contained r times in c, with no remainder: then it is asserted that d is the greatest common measure sought.

$$\begin{array}{l} b)\,a\,(p \\ \underline{pb} \\ \quad c)\,b\,(q \\ \quad \underline{qc} \\ \qquad d)\,c\,(r \\ \qquad \underline{rd} \\ \qquad\;\; 0 \end{array}$$

First, By the nature of division, $pb + c = a$, $qc + d = b$, and $rd = c$; then, since d measures c by the units

59. It is plain, that if the given quantities have a simple common measure, it may be rejected before the operation is begun; and that it must be afterwards introduced as a factor into the compound measure.

60. The greatest common measure of more than two quantities will be found by taking the G. C. M. of two of them, the G. C. M. of the result and a third, and so on whatever number there be. Thus, if A, B, C, D, be the quantities, find *m*, the G. C. M. of A and B, *n*, the G. C. M. of *m* and C, &c.; then because every common measure of A, B, and C measures *m* and C, and every measure of *m* and C measures A, B, and C, therefore *n*, the G. C. M. of *m* and C, must be the G. C. M. of A B, and C; and the same may be shewn of the others.

in r, it measures qc (Art. 54); and since it measures qc and d, it measures also $qc + d$ (Art. 55), that is, b; it measures b and therefore pb; it measures pb and c, and therefore $pb + c$, that is, a; but it was before shewn to be a measure of b, it is therefore a measure both of a and b.

Secondly, If a number measure a and b, it measures a and pb, and therefore their difference c (Art. 55); also it measures b and c, and therefore b and qc, and consequently their difference d; hence, *every number* which measures a and b measures d. It follows from this, that the *greatest number* which measures a and b, measures d. Now, no number greater than d can measure d, therefore no number greater than d can measure a and b; that is, d is the greatest common measure of a and b. Hence, to find the greatest common measure of two numbers, divide the greater by the less, and the last divisor by the last remainder, until there is no remainder: the last divisor is the measure required.

The greatest common measure of three numbers, a, b, c, is obtained by finding d the greatest common measure of a and b, and then the greatest common measure of d and c is the greatest common measure of a, b, c; because every common measure of a and b is a measure of d, and therefore the greatest common measure of d and c is the greatest common measure of a, b, c.

On the same principle, the greatest common measure of four or more numbers may be found.

EXAMPLES.

1. Required the G. C. M. of the quantities,

$$2b^3 - 10ab^2 + 8a^2b \text{ and } 9a^4 - 3ab^3 + 3a^2b^2 - 9a^3b.$$

The former of these is divisible by $2b$, the latter by $3a$ (1); this gives

$$b^2 - 5ab + 4a^2 \text{ and } 3a^3 - b^3 + ab^2 - 3a^2b.$$

Arranging the terms according to the powers of a (2), multiplying to avoid fractions (4), and then dividing, we have

$$\begin{array}{r@{}l}
 & 3a^3 - 3a^2b + ab^2 - b^3 \\
 & 4 \\
\cline{2-2}
4a^2 - 5ab + b^2) & 12a^3 - 12a^2b + 4ab^2 - 4b^3(3a \\
 & 12a^3 - 15a^2b + 3ab^2 \\
\cline{2-2}
 & \qquad 3a^2b + ab^2 - 4b^3
\end{array}$$

Dividing this remainder by b, and multiplying (4) the last divisor, which is the new dividend, by 3, which is not a factor of the other quantity, we find

$$\begin{array}{r@{}l}
3a^2 + ab - 4b^2) & 12a^2 - 15ab + 3b^2(4 \\
 & 12a^2 + 4ab - 16b^2 \\
\cline{2-2}
 & \qquad -19ab + 19b^2
\end{array}$$

Divide this remainder by $-19b$, and make the last divisor the dividend; the work will then terminate:

$$\begin{array}{r@{}l}
a - b) & 3a^2 + ab - 4b^2(3a + 4b \\
 & 3a^2 - 3ab \\
\cline{2-2}
 & \qquad 4ab - 4b^2 \\
 & \qquad 4ab - 4b^2 \\
\cline{2-2}
 & \qquad\quad 0
\end{array}$$

Hence, $a - b$ is the highest divisor required.

2. Required the G. C. M. of $36x^6 - 18x^5 - 27x^4 + 9x^3$ and $27x^5y^2 - 18x^4y^2 - 9x^3y^2$.

Here, $9x^3$ is obviously a factor of both quantities, and therefore (Art. 59) of the G. C. M. Dividing by $9x^3$, rejecting y^2, which is a factor of the latter quantity only, and, to avoid fractions, multiplying the dividend by 3, which is not a factor of the divisor, there is obtained, after division, the remainder,

$$2x^2 - 5x + 3 = 2x(x-1) - 3(x-1) = (2x-3)\,(x-1).$$

Rejecting $2x-3$, which is plainly not a factor of both quantities, $x-1$ is contained in the preceding divisor, without remainder. Hence, $9x^3(x-1)$ is the G. C. M. required.

3. Required the G. C. M. of $14x^3 + 7x^2 - 56x + 35$ and $7x^2 - 12x + 5$.

Employing the method of division by detached coefficients, given in Art. 45, the work will stand thus:

$$\begin{array}{r|r|l}
14 + 7 - 56 + 35 & 7 - 12 + 5 & -2 - 4 \\
-14 + 24 - 10 & -7 + 42 - 35 & -7 \\
-28 + 48 - 20 & & \\
\cline{1-2}
3 - 18 + 15 & 30 - 30 & \\
\text{or } 1 - 6 + 5 & \text{or } 1 - 1 & -1 + 5 \\
-1 + 1 & & \\
5 - 5 & & \\
\cline{1-1}
\pm 0 \pm 0 \pm 0 & &
\end{array}$$

Here placing the divisor to the right of the dividend, the quotient, with contrary signs, is $-2\ -4$, and the remainder divided by 3, gives $1-6+5$ for a new divisor. Dividing by this the first divisor, $7-12+5$, we have -7 for quotient, and $30-30$, or, by division, $1-1$, for the next divisor; dividing $1-6+5$ by it, the quotient is

$-1+5$, and there is no remainder. Hence $+1$, -1 are the coefficients of the G. C. M., it is therefore plainly $x-1$, since 12, 42, and 30, have x as their literal part.

4. Find the G. C. M. of the quantities $x^5-3ax^4+2a^2x^3+2a^3x^2-3a^4x+a^5$ and $x^5-ax^4+a^2x^3-a^3x^2-2a^4x+2a^5$.

$$\begin{array}{r|l|l}
1-3+2+2-3+1 & 1-1+1-1-2+2 & -1 \\
-1+1-1+1+2-2 & 2 & \\
\cline{1-2}
-2+1+3-1-1 & 2-2+2-2-4+4 & +1 \\
+2-10+6+10-8 & -2+1+3-1-1 & \\
\cline{1-2}
-9+9+9-9 & -1+5-3-5+4 & -2 \\
\text{or } 1-1-1+1 & +1-1-1+1 & +1-4 \\
 & -4+4+4-4 & \\
\cline{2-2}
 & \pm 0 \pm 0 \pm 0 \pm 0 \pm 0 &
\end{array}$$

Here, the terms being arranged as in last example, we have for the first remainder $-2+1+3-1-1$; the first divisor is doubled to render it divisible by this: subtracting we find $-1+5-3-5+4$, the first term of which is not divisible by 2; 4 is therefore annexed, and $-1+5-3-5+4$ made the new divisor. Dividing the preceding divisor by this, the next remainder is $-9+9+9-9$, which being divided by -9 gives $1-1-1+1$ for the next divisor; this is contained without remainder in the preceding divisor, $-1+5-3-5+4$, and the work terminates. Hence, $1-1-1+1$ represents the highest divisor, which by a comparison of the terms is easily seen to be $x^3-ax^2-a^2x+a^3$.

This example may also be worked by making the dividend divisor, and the divisor dividend, since both are of the same dimensions, and have unity for the coefficient of the first term; the work by this arrangement of the quantities will form perhaps a better exercise for the student than an additional example.

$$\begin{array}{r|r|l}
1-1+1-1-2+2 & 1-3+\ 2+\ 2-\ 3+1 & -1 \\
-1+3-2-2+3-1 & 2 & \\
\cline{1-2}
2-1-3+1+1 & 2-6+\ 4+\ 4-\ 6+2 & -1+5 \\
-2+2+2-2 & -2+1+\ 3-\ 1-\ 1 & \\
\cline{2-2}
-1+1+1-1 & -5+\ 7+\ 3-\ 7+2 & \\
 & 2 & \\
\cline{1-2}
\pm0\pm0\pm0\pm0\pm0 & -10+14+\ 6-14+4 & \\
 & 10-\ 5-15+\ 5+5 & \\
\cline{2-2}
 & 9-\ 9-\ 9+9 & \\
 & \text{or, } 1-\ 1-\ 1+1 & -2-1
\end{array}$$

The divisor being placed to the right of the dividend as before, the first quotient is − 1, and remainder 2 − 1 − 3 + 1 + 1; the first divisor is then doubled to render it divisible by this remainder; the division being then performed, it is seen, by performing the addition mentally, that the first term of the remainder would be − 5, and therefore not divisible by the first term, 2, of the divisor; the full remainder is therefore brought down and doubled, and the division proceeded with; the remainder divided by 9 gives 1 − 1 − 1 + 1 for the new divisor; by this the preceding divisor 2 − 1 − 3 + 1 + 1 is divided, and there is no remainder. The G. C. M. then is, as before, $x^3 - ax^2 - a^2x + a^3$.

5. Required the highest divisor of the expressions $x^6 + x^2y - x^4y^2 - y^3$ and $x^4 - x^2y - x^2y^2 + y^3$.

Here $x^6 + x^2y - x^4y^2 - y^3 = (x^2 - y^2)(x^4 + y)$, and $x^4 - x^2y - x^2y^2 + y^3 = (x^2 - y^2)(x^2 - y)$; and the highest divisor is at once seen to be $x^2 - y^2$, since $x^4 + y$ and $x^2 - y$ have obviously no divisor in common. Several of the exercises which follow may be worked in this way more readily than by the ordinary method. A little consideration, or a few trials, will discover the component factors, for no rule can be given for such resolution.

EXERCISES.

Find the highest common divisors of the following quantities :—

1. $x^3 - a^3$ and $x^2 - a^2$.

2. $a^4 - x^4$ and $a^3 + a^2x - ax^2 - x^3$.

3. $6a^2 + 11ax + 3x^2$ and $6a^2 + 7ax - 3x^2$.

4. $15a^5 + 10a^4b + 4a^3b^2 + 6a^2b^3 - 3ab^4$ and $12a^3b^2 + 38a^2b^3 + 16ab^4 - 10b^5$.

5. $3x - 5 - 18x^2 + 4x^4 + 2x^3$ and $- 11x^3 - 3x^2 - 1 - 3x + 6x^5 - 4x^4$.

6. $y^2 + 5y + 4$, $y^2 + 2y - 8$ and $y^2 + 7y + 12$.

7. $ax^2 - a^2x + 14x^3 + a^3$ and $4ax^2 - 2a^2x + 7x^3 + a^3$.

8. $15a^4 - 9a^3 + 47a^2 - 21a + 28$ and $20a^6 - 12a^5 + 16a^4 - 15a^3 + 14a^2 - 15a + 4$.

9. $x^4 + 2x^2 + 9$ and $7x^3 - 11x^2 + 15x + 9$.

10. $9a^6 + 11a^4 + 18a^3 + 42a^2 - 8$ and $18a^6 - 5a^4 - 18a^3 + 33a^2 - 4$.

11. $x^6 + 4x^5 - 3x^4 - 9 + 12x + 11x^2 - 16x^3$ and $6x^5 - 12x^3 + 12 - 48x^2 + 20x^4 + 22x$.

12. $x^4 - px^3 + (q - 1)x^2 + px - q$ and $x^4 - qx^3 + (p - 1)x^2 + qx - p$.

13. $12x^4 - 40ax^3 + 28a^2x^2$ and $2x^4 - 10ax^3 + 14a^2x^2 - 6a^3x$.

14. $x^2 - 7x + 10$ and $4x^3 - 25x^2 + 20x + 25$.

15. $8x^4 - 30x^3 + 31x - 12$ and $16x^4 - 53x^2 + 45x + 6$.

16. $2x^6 - 11x^3 - 9$ and $4x^5 + 11x^4 + 81$.

17. $4adq - 42xy + 24ad - 7qxy$ and $3bcq + 30mp + 18bc + 5mpq$.

18. $a^2b^2 - b^2x^2 - a^2x^2 + x^4$ and $4a^2b - 2ax^2 + 2x^3 - 4abx$.

19. $a^2 - \frac{a + 1}{4}$ and $\frac{1}{3}a^2 + \frac{1}{6}\, 11a\sqrt{(a + 1)} - a - 1$.

20. $6ax^2 - 6ab^2 - 54a - 21x^2 + 21b^2 + 189$; $6ax^5 + 60a - 21x^5 - 210$; and $6ab^3 - 12a - 21b^3 + 42$.

61. To find the least common multiple of two quantities,—*Divide their product by their highest common divisor.*

Let a and b be the quantities, and x their highest divisor; and let p and q be the quotients arising from dividing a and b by this divisor, so that $\frac{a}{x}=p$, $\frac{b}{x}=q$; then, since x is the *highest possible* divisor, p and q have no common divisor.* Hence, pq is the least common multiple of p and q, and therefore pqx is the least common multiple of px and qx. Now, $px=a$, and $qx=b$, therefore $ab=px \times qx$; whence, $\frac{ab}{x}=pqx$. Hence, $\frac{ab}{x}$, or $\frac{a}{x} \times b=pb$, is the least common multiple of a and b; that is, the L. C. M. is the product divided by the common divisor; or, the product of one of the quantities by the quotient of the other and the common divisor.

62. It is plain, that the least common multiple of two incommensurable quantities is their product.

63. The least common multiple of several quantities will be found by finding the least common multiple of the first two, the least common multiple of that multiple and the third, and so on.

64. Since dividing quantities by their highest common divisors is the same as detaching all the common factors, thus leaving results which are prime to one another, it is

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* For, if they had a common divisor, it would enter as a factor into the divisor of a and b, and there would therefore be a divisor greater than x; that is, we should have $\frac{a}{x}=np$, and $\frac{b}{x}=nq$; whence $\frac{a}{nx}=p$, and $\frac{b}{nx}=q$, which is contrary to the hypothesis.

plain that the least common multiple of several quantities will be *the continued product of the common measures which exist between two or more of them, and the quotients obtained by dividing them by all the common factors.* Thus, if the quantities be $4a$, $6ab$, $8abc$, we first detach the factor $2a$, the G. C. M., giving 2, $3b$, $4bc$, then the factor b, giving 2, 3, $4c$, and lastly the factor 2, giving 1, 3, $2c$; then $2a \times b \times 2 \times 1 \times 3 \times 2c = 24abc$, the least common multiple; the continued product of the quantities being $192a^3b^2c$. From this the reason of the common arithmetical process is manifest.

EXAMPLES.

1. Required the L. C. M. of the quantities $6a^2 + 11ax + 3x^2$ and $6a^2 + 7ax - 3x^2$.

The G. C. M. of these quantities is $2a + 3x$ (Ex. 3, Art. 60), hence the quotient of the first quantity and this measure is $3a + x$; then $(6a^2 + 7ax - 3x^2)(3a + x) = 18a^3 + 27a^2x - 2ax^2 - 3x^3$ is the L. C. M. required.

2. Required the L. C. M. of 325 and 455.

The G. C. M. of these numbers is 65; then $325 \div 65 = 5$; therefore $455 \times 5 = 2275$ is the L. C. M. required.

EXERCISES.

Find the least common multiple of the following quantities:—

1. $8x^2$, $12x^3$ and $20x^4$.
2. $x - 1$, $x + 1$ and $x^2 - 1$.
3. $6ax$, $15bx$, $27b$ and $35abx$.
4. $2x^3 - 5x^2 + 5x - 3$ and $8x^2 - 10x - 3$.

5. $x^3 - a^3$ and $x^2 - a^2$.

6. $x^3 - 3x^2 + 7x - 21$ and $x^4 - 49$.

7. $a^4 - x^4$ and $a^3 - a^2x - ax^2 + x^3$.

8. $x^5 + ax^4 + a^2x^3 + a^3x^2 + a^4x + a^5$ and $x^5 - ax^4 + a^2x^3 - a^3x^2 + a^4x - a^5$.

9. $3x^2 - 11x + 6$, $2x^2 - 7x + 3$ and $6x^2 - 7x + 2$.

10. $x^4 - 2x^3 + 2x^2 - 2x + 1$, $x^4 - 2x^3 + 2x - 1$, $x^3 - 3x^2 + 3x - 1$ and $x^3 - x^2 - x + 1$.

VII. FRACTIONS.

65. In Algebra, the term *fraction* is not applied as in Arithmetic, to denote merely the quotient of a *less* quantity by a *greater*; but the division of any quantity by another, whether greater than it or not, when the division is *indicated* only, *not performed*. Thus $\frac{a}{b}$ is a fraction, denoting that the quantity a is to be divided by b, without reference to a being greater or less than b, that is, to the quotient being expressed in units, or parts of units; it merely indicates that the bth part of a is to be taken. Now, since $a = 1 \times a$, that is, since a is supposed to be a times greater than 1, the bth part of a will be a times greater than the bth part of 1; that is, $\frac{a}{b}$ is a times greater than $\frac{1}{b}$, or $\frac{a}{b} = \frac{1}{b} \times a$. Hence $\frac{a}{b}$ denotes also that a unit is divided into b equal parts, and a of those parts taken. The fraction $\frac{a}{b}$ may therefore be named either, the bth part of a; or, a times the bth part of 1. It is in this latter view that the dividend a is called the *numerator*, and the divisor b the *denominator*, as in Arithmetic; when spoken of together, they are called the *terms* of the fraction.

66. *A fraction is multiplied by any quantity when its numerator is multiplied, or its denominator divided, by that quantity.*

Thus $\frac{a}{b} \times c = \frac{ac}{b}$; for, as explained above, there are c times as many parts taken in the fraction $\frac{ac}{b}$ as in the fraction $\frac{a}{b}$, the unit being divided into the same number of equal parts in both.* Again, $\frac{a}{bc} \times c = \frac{a}{b}$; because in each of the fractions $\frac{a}{bc}$ and $\frac{a}{b}$ the numerators are the same, that is, the same number of parts is taken, but the parts are c times greater in the latter, the unit being divided into b parts only; whereas in the former it is divided into bc parts, and each part is therefore $\frac{1}{c}$th part of $\frac{a}{b}$; that is, $\frac{a}{b}$ is c times greater than $\frac{a}{bc}$.

67. *A fraction is divided by any quantity when its numerator is divided, or its denominator multiplied, by that quantity.*

Let $\frac{ac}{b}$ be a fraction, and c a whole number; then $\frac{ac}{b} \div c = \frac{a}{b}$; for in both fractions, $\frac{ac}{b}$ and $\frac{a}{b}$, the unit is divided into the same number of equal parts, namely b, and in the former there are c times more of these parts taken than in the latter; therefore $\frac{a}{b}$ is c times less than $\frac{ac}{b}$; that is, the fraction is divided by dividing its numerator.

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* The quantity c is here supposed to be integral, as are also the quantities a and b. But the letters may represent fractions as well as whole numbers. It is necessary in all cases, both in following out reasonings, and in presenting results, to use the phraseology which belongs to whole numbers—our language hardly admitting of any other; in considering fractions, the terms which we employ, such as multiplication and division, require but a slight modification or extension of meaning. It is manifest from what we have repeatedly seen already, that the letters which we employ may represent fractions; while such expressions as $\frac{a}{b}$, $\frac{c}{d}$ may be integral.

Again, let the fraction be $\frac{a}{b}$; then $\frac{a}{b} \div c = \frac{a}{bc}$, because in this latter, the unit is divided into c times as many parts as in the former, and each part is therefore c times less; but the same number of parts in each is taken; so that the fraction $\frac{a}{bc}$ is c times less than $\frac{a}{b}$, and hence to divide a fraction we multiply its denominator.

68. *If the terms of a fraction be multiplied or divided by the same quantity, the value of the fraction is not changed.*

Let $\frac{a}{b}$ be a fraction, and c a whole number; then, if the numerator is multiplied by c, the fraction is multiplied, and becomes $\frac{ac}{b}$ (Art. 66); and if the denominator of this be multiplied by c, the fraction is divided by the same quantity (Art. 67), and becomes $\frac{ac}{bc}$; hence its value is not altered. By the multiplication of the numerator, the fraction is multiplied by c, and by the multiplication of the denominator the fraction is divided by the same quantity, so that $\frac{ac}{bc}$ is of the same value as $\frac{a}{b}$. Again, let the fraction be $\frac{ac}{bc}$; then, if the denominator be divided by c, giving $\frac{ac}{b}$, the fraction is multiplied (Art. 66) by c, or increased c times, while if the numerator be divided by c, the fraction is divided by c (Art. 67), or its value is diminished the same number of times; hence, $\frac{a}{b}$ is of the same value as $\frac{ac}{bc}$.

We may take the other view of a fraction, regarding a as a dividend, and b as a divisor, and the truth of the proposition will equally appear; for it is a well-known principle in division that a quotient is not altered, if the divisor and dividend be either both increased or both diminished, the same number of times.[*]

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[*] The three important principles above stated, on which the whole theory of fractions rests, may be otherwise proved as follows:—

Let $\frac{a}{b}$ be a fraction, and let f express this quotient, so that $f=\frac{a}{b}$. Then by the nature of division, $fb=a$; whence (Art. 29), $mfb=ma$, and (Art. 30) $f=\frac{ma}{mb}$; wherefore $\frac{a}{b}=\frac{ma}{mb}$; and thus if we multiply the terms of $\frac{a}{b}$ by m, we obtain $\frac{ma}{mb}$, while if we divide the terms of $\frac{ma}{mb}$ by m, we obtain $\frac{a}{b}$; from which the truth of the proposition stated in Art. 67 appears. Again, from the expression $mfb=ma$, we have, dividing by b, $mf=\frac{ma}{b}$, or $m\frac{a}{b}=\frac{ma}{b}$; so that, multiplying the numerator multiplies the fraction. Comparing the expressions $mf=\frac{ma}{b}$, and $f=\frac{ma}{mb}$, we see that the former is obtained from the latter by dividing the denominator and multiplying f by m; so that dividing the denominator also multiplies the fraction. Lastly, since $fb=a$, if we divide these equals by mb, we find $\frac{f}{m}=\frac{a}{mb}$. Comparing this with the expressions $f=\frac{ma}{mb}$, and $f=\frac{a}{b}$, we see that it is obtained if, in the first, we divide f and the numerator by m; or if, in the second, we divide f and multiply the denominator by m; so that the value of the fraction f is divided by dividing the numerator, or multiplying the denominator.

These and similar proofs are unsatisfactory and misleading, by reason of the first assumption, that the fraction has a value expressible by a distinct quantity, whether f, x, or any other. An irreducible fraction has no such value, unless we change the unit of measure; such fractions as $\frac{2}{3}$, $\frac{7}{11}$, $\frac{5}{8}$ cannot be expressed by any other number or quantity whatever, and yet the quantity $\frac{a}{b}$ may stand for such fractions. Hence a proof, though having less of an analytical character, founded on our first notions of the nature of a fraction, seems preferable.

69. To reduce a fraction to its lowest terms. *Find the greatest common measure of the terms* (Art. 58), *and divide both the terms by it.**

It follows from the nature of the greatest common measure, that when both terms of a fraction are divided by it, they are rendered prime to one another, and the fraction is therefore irreducible; the reduced fraction is of the same value as the original one (Art. 67).

Ex. 1. Reduce $\frac{24a^2x^3y^2}{8abx^4}$ to its lowest terms.

Here $8ax^3$ is plainly the highest divisor of both terms; dividing by this quantity, we find $\frac{3ay^2}{bx}$ for the fraction in its simplest form.

Ex. 2. Reduce the fraction $\frac{2x^3 - 16x - 6}{3x^3 - 24x - 9}$ to its lowest terms.

Here, since 2 is a factor of all the terms of the numerator, and 3 of those of the denominator, it is plain that

$$\frac{2x^3 - 16x - 6}{3x^3 - 24x - 9} = \frac{2}{3}\left(\frac{x^3 - 8x - 3}{x^3 - 8x - 3}\right) = \frac{2}{3}.$$

Ex. 3. Reduce $\frac{15x^3 + 35x^2 + 3x + 7}{27x^4 + 63x^3 - 12x^2 - 28x}$ to its lowest terms.

The greatest common measure must, in this case, be found by Art. 58. The work is as follows:

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* Or, find by inspection any common divisor of the terms, and divide them both by it; the fraction is thus reduced to *lower* terms; repeat the process till no common divisor of the terms can be found, and the fraction will then be in its *lowest* terms.

$$\begin{array}{r|r|l}
27 + 63 - 12 - 28 & & \\
5 & 15 + 35 + 3 + 7 & -9 \\ \cline{1-1}
135 + 315 - 60 - 140 & -15 - 35 & \\
-135 - 315 - 27 - 63 & -3 - 7 & -5 - 1 \\ \cline{1-2}
-87 - 203 & \pm 0 \pm 0 \pm 0 \pm 0 & \\
\text{or, dividing by } -29,\ 3 + 7 & &
\end{array}$$

Whence the last divisor, $3x + 7$, is the greatest common measure; dividing both terms of the original fraction by it, we obtain, for the reduced fraction,

$$\frac{5x^2 + 1}{9x^3 - 4x}.$$

70. To reduce an integer to the form of a fraction, having a given denominator,—*Multiply the integer by the given denominator, and under the product, taken as a numerator, set that denominator.*

This is merely performing a process and indicating its reverse: and therefore the value of the quantity is not changed. Thus, to express a by a fraction, whose denominator shall be $b - x$, we perform the multiplication, and indicate the division, and thus have $\frac{ab - ax}{b - x}$ or $\frac{a(b - x)}{b - x}$; and to put $x - 3$ under a fractional form, so that the denominator may be the same as that of the expression $\frac{5a}{n + y}$, we have $\frac{x(n + y) - 3(n + y)}{n + y}$, or $\frac{nx + ny - 3n - 3y}{n + y}$.

71. *By performing the division which is merely indicated,* a fractional expression may often be reduced to an integral or mixed quantity.

Thus, $\frac{8b^2x - 4ab^2 - 3a^2b}{4b^2}$ is reduced by actual division to $2x - a - \frac{3a^2}{4b}$.

72. Fractions having different denominators will be reduced to equivalent ones having a common denominator, by *multiplying both terms of each fraction by all the denominators except its own.*

Since both terms of each fraction are multiplied by the same quantity, the new fractions are respectively equal (Art. 68) to the original ones; they must all have the same denominator, because each denominator is the product of the same factors.

Reduce $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$, to fractions having a common denominator.

Here (Art. 68), $\frac{a}{b} = \frac{adf}{bdf}, \frac{c}{d} = \frac{cbf}{bdf}$, and $\frac{e}{f} = \frac{ebd}{bdf}$.

73. When the fractions are such that the denominators have a common multiple less than their continued product, the process of finding the common denominator may be much simplified, by applying the following rule :—

Find the least common multiple of the denominators (Art. 61), *and multiply both terms of each fraction by the quotient of the common multiple and the denominator of that fraction.*

Ex. 1. Let it be required to reduce the fractions

$$\frac{a}{px}, \frac{b}{qx}, \frac{c}{ry}, \frac{d}{sy},$$

to equivalent fractions having a common denominator.

Here, the least common multiple of the denominators is $pqrsxy$, and this, divided by the denominators in succession, gives the quotients

$$qrsy,\ prsy,\ pqsx,\ pqrx\ ;$$

whence, multiplying both terms by these quotients respectively, we obtain for the reduced fractions

$$\frac{aqrsy}{pqrsxy}, \frac{bprsy}{pqrsxy}, \frac{cpqsx}{pqrsxy}, \frac{dpqrx}{pqrsxy}.$$

With respect to the reason of this, it is plain, that by using the least common multiple instead of the continued product, those factors are omitted which would be common to both terms of all the reduced fractions; and that the division of the common multiple by each denominator, and the multiplication of both terms by the quotient, will reproduce the common multiple each time for a denominator.

Ex. 2. Reduce $\frac{5a}{4bd}, \frac{2c}{3b}, \frac{7x}{6d}$, to a common denominator.

Here, the least common multiple is $12bd$; and $12bd$ divided by each of the denominators gives 3, $4d$, $2b$: when both terms are multiplied by these, we find $\frac{15a}{12bd}, \frac{8cd}{12bd}, \frac{14bx}{12bd}$ for the reduced fractions. In most cases, however, we can easily find by inspection the quantity by which both terms are to be multiplied.

74. To add or subtract fractions,—*Reduce them, if necessary, to equivalent ones having a common denominator, and under the sum or difference of the numerators, place the common denominator.*

Let it be required to find the sum and difference of the fractions $\frac{a}{b}$ and $\frac{c}{d}$.

The equivalent fractions are (Art. 72) $\frac{ad}{bd}$ and $\frac{cd}{bd}$. Now, by the nature of division, the sum or difference of the quotients of any quantities by the same divisor, is equal to the quotient of the sum or difference of those quantities by that divisor; and since the reduced fractions are equivalent to the original ones, it follows that

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad}{bd} \pm \frac{bc}{bd} = \frac{ad \pm bc}{bd}.$$

It follows from the very nature of a fraction (Art. 65), that fractions having different denominators cannot be incorporated into one sum; they can only be combined by means of their signs.

75. A mixed quantity is reduced to a fractional form by *expressing the integral part as a fraction with the same denominator as the fractional part* (Art. 70), *and then proceeding as directed in the last rule.*

Thus, to reduce $a \pm \frac{b}{c}$ to a fractional form, we make $a = \frac{ac}{c}$ (Art. 70); then $a \pm \frac{b}{c} = \frac{ac}{c} \pm \frac{b}{c} = \frac{ac \pm b}{c}$. And to reduce $a - \frac{b-c}{d}$ to a fractional form, we have $a = \frac{ad}{d}$; therefore,

$$a - \frac{b-c}{d} = \frac{ad}{d} - \frac{b-c}{d} = \frac{ad-(b-c)}{d} = \frac{ad-b+c}{d}.$$

76. To multiply fractions,—*Multiply the numerators together for a numerator, and the denominators for a denominator.*

A fraction may be multiplied by an integer, either by expressing the integer as a fraction (Art. 70), and applying the above rule, or by applying Art. 66.

EXAMPLES.

1. Let it be required to multiply $\frac{a}{b}$ by $\frac{c}{d}$.

This denotes that the fraction $\frac{a}{b}$ is to be divided into d equal parts, and the quotient repeated c times, or that c such parts are to be taken. First, then, dividing by d, we have (Art. 67) $\frac{a}{bd}$ for one of the equal parts; taking c of these parts, or repeating the quotient c times (Art. 66), gives $\frac{ac}{bd}$.

The process may often be shortened by suppressing, while performing the multiplication, those factors which are common to the numerator and denominator: as this is, in fact, anticipating the reduction of the resulting fraction to its lowest terms.*

2. Multiply together the fractions $\frac{nx}{3p}, \frac{2a}{y}, \frac{3b}{2a}$.

Here, multiplying the three numerators continually together, and also the three denominators, and at the same time suppressing the common factors $2a$ and 3, we obtain $\frac{bnx}{py}$.

3. Multiply $\frac{3a}{12bx}$ by $3b$.

Here the denominator is divisible by the integer, therefore (Art. 65) the product is $\frac{3a}{4x}$, which is irreducible. If we had multiplied the numerator, we should have had to reduce by dividing both terms by $3b$. If $3b$ be expressed as a fraction (Art. 70), we shall have,

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* The reason of the rule may be shewn otherwise thus: let $f = \frac{a}{b}$, $f' = \frac{c}{d}$; then multiplying these by b and d, we have $bf = a$ and $df' = c$. Whence by multiplication we find $bdff' = ac$; and dividing by bd, we have for the product of the two fractions, $ff' = \frac{ac}{bd}$

$$\frac{3a}{12bx} \times \frac{3b}{1} = \frac{9ab}{12bx} = \frac{3a}{4x}.$$

which is the same as multiplying the numerator by the integer.

4. Multiply $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{6} + \frac{x}{8}$ by the least common multiple of the denominators.

The multiple is plainly 24; multiplying the numerators by this number, and performing the actual division by the denominators (Art. 71), we obtain $+ 12x + 8x + 6x + 4x + 3x = 33x$.

77. To divide by a fraction,—*Invert the divisor, and proceed as in multiplication.**

A fraction may be divided by an integer, either by expressing the integer as a fraction (Art. 70), and applying the above rule, or by applying Art. 67.

Let it be required to divide $\frac{a}{b}$ by $\frac{c}{d}$

The fraction $\frac{c}{d}$ denotes either that the quantity c is divided into d equal parts, or that unity is divided into d such parts, and c of them taken; that is, it denotes either the dth part of c, or c times the dth part of 1. In either view the quantity c is d times greater than the fraction $\frac{c}{d}$. Hence if we divide $\frac{a}{b}$ by c, obtaining (Art. 67) $\frac{a}{bc}$, we get a quotient which is d times too little; this quotient must therefore be multiplied by d, which gives $\frac{ad}{bc}$, (Art. 66). Now $\frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c}$, or $\frac{a}{b}$ multiplied by the divisor inverted.*

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* If both divisor and dividend be fractions, and if each term of the divisor measure the corresponding term of the dividend, the quotient may be obtained by dividing the numerator and denominator of the dividend by the numerator and denominator of the divisor, respectively. Thus $\frac{15ax^2}{32b^3y} \div \frac{5ax}{8b^2y} = \frac{3x}{4b}$. This rule is, however, so seldom applicable, that it is of little use.

78. Hence, unity divided by a fraction is the same as that fraction inverted; for $1 \div \frac{b}{a} = 1 \times \frac{b}{a} = \frac{b}{a}$.

79. Unity divided by any quantity is called the *reciprocal* of that quantity: thus $\frac{1}{x}$ is the reciprocal of x, and $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$. Since an integral quantity may be considered as a fraction whose denominator is 1, the reciprocal of any quantity, integral as well as fractional, may be considered as that quantity inverted.

80. Hence, to divide by any quantity is the same as to multiply by its reciprocal, and *vice versa*.

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* The same may be proved otherwise, thus:—1. By the nature of division, the quotient must be such, that, when multiplied by the divisor, it shall reproduce the dividend. Calling the quotient in this case x, we must have $\frac{a}{b} = \frac{c}{d} \times x = \frac{cx}{d}$; multiplying these equals by d, we have $\frac{a}{b} \times d = \frac{cx}{d} \times d$, (Art. 29), or $\frac{ad}{b} = cx$; then dividing these equals (Art. 30) by c, so as to obtain an expression for x, the quotient, we find $\frac{ad}{bc} = x$. Now $\frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c}$; therefore, &c.

2. Let $f = \frac{a}{b}$ and $f' = \frac{c}{d}$; then $fb = a$, and $f'd = c$. Multiply the first of these by d, and the second by b, and we have $fbd = ad$ and $f'bd = bc$. Whence by division $\frac{fbd}{f'bd} = \frac{ad}{bc}$, or $\frac{f}{f'} = \frac{ad}{bc}$, as before.

EXAMPLES.

1. Divide $\frac{n^2-9}{x-1}$ by $\frac{n+3}{x^2-2x+1}$.

Here inverting the divisor, indicating the multiplication, and then resolving into factors, we have,

$$\frac{(n+3)(n-3)}{x-1} \times \frac{(x-1)^2}{n+3}.$$

Then, cancelling the common factors, we have for the result $(n-3)(x-1)$.

2. Divide $a - \frac{b}{c}$ by $x + \frac{d}{y}$.

Reducing these to a fractional form (Art. 75), and dividing, we find,

$$\frac{(ac-b)y}{(xy+d)c}.$$

3. Simplify the expression,

$$\frac{1}{x + \frac{1}{1 + \frac{x+1}{3-x}}}$$

Here the complex denominator is first reduced to a fractional form, and then 1 is divided by it; this gives,

$$\frac{1}{x + \frac{3-x}{4}}.$$

Which farther reduced in the same manner, gives $\frac{4}{3(x+1)}$.

EXERCISES IN FRACTIONS.

1. $\frac{1}{b} \times b$
2. $\frac{3}{11}a \times 11$
3. $\frac{2}{3}ax \times 3c.$
4. $\frac{1}{2}a \times 4$
5. $\frac{3}{4}b \times 4ax$
6. $\frac{5bx}{4a} \times 4a$
7. $\frac{1}{y} \div x$
8. $\frac{1}{2} \div a$
9. $\frac{3}{2}n^2x \div 3x$
10. $\frac{3}{4}a \div 3b$
11. $\frac{7}{8}a \div 3a$
12. $\frac{a}{b} \div c$
13. $\frac{n-1}{n+1} \times (n+1)$
14. $\frac{2ay}{15n^2x} \times 5n$
15. $\frac{a}{a^2-x^2} \times (a-x)$
16. $\frac{9a^2bd}{15abcd} \div 3ad$
17. $\frac{m^2-q^2}{(m+q)^2} \div (m+q)$
18. $\frac{n^2-9}{a+3} \div (n-3)$
19. $\frac{1}{b-x} \div (b-x)$
20. $\frac{\frac{1}{1-x}}{4} \times (1-x)$

Reduce to lowest terms,

21. $\frac{9abx}{12a^2by}$
22. $\frac{2a^3x^3}{5a^2x^2b}$
23. $\frac{4a^2x^3y}{12a^3x^3b}$
24. $\frac{a^n}{a^{n+1}}$
25. $\frac{ax+x^2}{3bx-cx}$
26. $\frac{16bc^3x^2}{20ax^3c^2}$
27. $\frac{n^2-9}{n^2-6n+9}$
28. $\frac{a^3-b^3}{a^2-b^2}$
29. $\frac{a^3-b^3}{(a-b)^2}$
30. $\frac{n^3-2n^2}{n^2-4n+4}$
31. $\frac{14a^2-7ab}{10ac-5bc}$
32. $\frac{x^2+2x-3}{x^2+5x+6}$

33. $\dfrac{a^3x - 2a^2x^2 + ax^3}{a^4x + a^3x^2 + a^2x^3 + ax^4}$

34. $\dfrac{10x^2 + 11x - 6}{15x^3 - 6x^2 - 35x + 14}$

35. $\dfrac{6a^4 - 5a^2x^2 - 6x^4}{4a^5 - 6a^3x^2 - 2a^2x^3 + 3x^5}$

36. $\dfrac{x^{2n+1} + a^{2n+1}}{x^{2m+1} + a^{2m+1}}$

37. $\dfrac{x^3 - 39x + 70}{x^2 - 3x - 70}$

38. $\dfrac{3a^3 - 3a^2b + ab^2 - b^3}{4a^2 - 5ab + b^2}$

39. $\dfrac{x^4 - 3x^3 + x^2 + 3x - 2}{4x^3 - 9x^2 + 2x + 3}$

40. $\dfrac{x^3 - x^2 - 21x + 45}{3x^2 - 2x - 21}$

41. $\dfrac{21a^3b^2x - 9ab^3x^2}{15a^2b^2x + 3a^5b^4x^2 - 12ab^2x}$

42. $\dfrac{10a^nb^3c^2 - 15a^{n+1}b^2c^3}{20a^{n+}b^4c^2 - 5a^nb^2c^2}$

43. $\dfrac{x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6}{3x^5 - 10x^3 + 15x + 8}$

44. Express $3x$ as a fraction with the denominator $5b$

45. ,, $a-x$,, ,, ,, $a+x$

46. ,, $12a^2x$,, ,, ,, $5y^2$

47. ,, $2a-1$,, ,, ,, $a-1$

48. ,, $\sqrt{x}$,, ,, ,, a

49. ,, $(a-x)^3$,, ,, ,, of $\dfrac{b}{(a-x)^2}$

Reduce to integral or mixed quantities,

50. $\dfrac{b(a^2 - x^2)}{a - x}$

51. $\dfrac{1}{2c}(8c^2 + 6c + 5)$

52. $\dfrac{1}{3b}(6b^2x - 9ab + 21)$

53. $\dfrac{a(b^5 + x^5) - c}{b + x}$

54. $\dfrac{1}{x-y}(x^3 - y^3 + x^2 - 2y^2)$

55. $\dfrac{1 - b^2 - a(1-b)}{1 - b}$

56. $4ax^2 \cdot 6b^2x^2 + 2x^3 - 10bx^2$

57. $\dfrac{3a^5 + 16a^4b - 33a^3b^2 + 14a^2b^3}{a^2 + 7ab}$

Reduce to a common denominator

58. $\frac{2x+1}{b^2x}$ and $\frac{3a-1}{x}$

59. $\frac{a}{6xy}, \frac{b}{3x}$ and $\frac{c}{2y}$

60. $d, \frac{x+1}{3}$ and $\frac{n}{m}$

61. $\frac{x}{a+b}, \frac{y}{a-b}$ and $\frac{z}{a^2-b^2}$

62. $\frac{6a-3}{5ax}$ and $\frac{3y}{5a}$

63. $\frac{p-q}{a^2-n^2}$ and $\frac{x}{a+n}$

Find the sum or difference, according to the sign:—

64. $\frac{a}{b}+\frac{x}{b}$

65. $\frac{a}{3}+\frac{a}{4}$

66. $\frac{a}{b} \pm \frac{3a}{4b}$

67. $\frac{m}{n} \pm \frac{p}{q}$

68. $\frac{m}{n} \pm \frac{n}{m}$

69. $\frac{a+x}{a-x} \pm \frac{a-x}{a+x}$

70. $\frac{1}{1+x} \pm \frac{1}{1-x}$

71. $\frac{5x}{7a} - \frac{3y}{7}$

72. $\frac{3x}{7} - \frac{5x}{14}$

73. $\frac{4x}{5} \pm \frac{3x}{4}$

74. $\frac{p+q}{p-q} - \frac{p-q}{p+q}$

75. $\frac{n}{n-1} - \frac{n-1}{n}$

76. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

77. $\frac{a}{10} + \frac{b}{10^2} + \frac{c}{10^3}$

78. $\frac{1}{a-b} - \frac{1}{a+b}$

79. $\frac{1}{1-x} - \frac{2}{1-x^2}$

80. $\frac{4a}{b} - \frac{4a-x}{b}$

81. $\frac{1}{x-6} \pm \frac{1}{x-5}$

82. $\frac{1}{3(1 \mp x)} + \frac{2 \pm x}{3(1 \pm x + x^2)}$

83. $\frac{1}{x-2} - \frac{1}{x-5}$

84. $\frac{n_1}{x+a_1} + \frac{n_2}{x+a_2} + \frac{n_3}{x+a_3}$

85. $\frac{1}{2x+2} - \frac{4}{x+2} + \frac{9}{2(x+3)}$

86. $\frac{3+2x}{2-x} - \frac{2-3x}{2+x} + \frac{16x-x^2}{x^2-4}$

87. $\frac{1}{3}(2a+1)+\frac{1}{5}(4a+2)+\frac{1}{7}a$

88. $\frac{1}{4}(a-3x)+\frac{1}{5}(3a-5x)+\frac{1}{20}(3a-5x)$

89. $\frac{a+b}{(b-c)(c-a)}+\frac{b+c}{(c-a)(a-b)}+\frac{c+a}{(a-b)(b-c)}$

Reduce to a fractional form and simplify

90. $a-7-\frac{-50+2a^2}{a+7}$

91. $5x-2y^2+\frac{4xy-3y^3}{3y}$

92. $2x-\frac{5}{12}x$

93. $a+b+\frac{5a-2b^2}{2b}$

94. $x-y-\frac{x^2-y^2-5}{x+y}$

95. $a-\frac{3}{4}\cdot\frac{a}{b}$

96. $b+y-\frac{b^2-y^2-2}{b-y}$

97. $\left(1+x+x^2+\frac{x^3}{1-x}\right)-\left(1+x+\frac{x^2}{1-x}\right)$

98. $5ax-3by-\frac{10a^2x-6aby}{2a}$

99. $m-x-\frac{m^2-2mx+6+x^2}{m-x}$

Multiply

100. $\frac{a+b}{2x}$ by $\frac{a-b}{3a}$

101. $\frac{a}{6}$ by $\frac{2a^2}{9}$

102. $\frac{2}{3}a$ by $\frac{3}{4}b$

103. $\frac{b}{c}$, $\frac{x}{a}$ and $\frac{a}{y}$

104. $\frac{a+x}{a-x}$ by $\frac{a \quad x}{a+x}$

105. $\frac{3q^2-q}{5}$ by $\frac{10}{2q^2-4q}$

106. $\frac{a^2+b^2}{a^2-b^2}$ by $\frac{a-b}{a+b}$

107. $\frac{x^2-1}{b^2-16}\times\frac{b-4}{x+1}$

108. $a+\frac{1}{a}$ by $a-\frac{1}{a}$

109. $\frac{3a}{4x}+b$ by $\frac{5a}{3x}-2b$

110. $\frac{1}{3}-3a$ by $\frac{2}{3}-6a$

111. $1-\frac{a-x}{a+x}$ by $2+\frac{2x}{a-x}$

112. $\dfrac{a^n - x^n}{4a - 2}$, $\dfrac{a - x}{b^n + x^n}$ and $\dfrac{2a - 1}{b^n - x^n}$

113. $\dfrac{a^2 - x^2}{c^2 - x^2}$, $\dfrac{bc + bx}{a^2 + ax}$, $\dfrac{c - x}{a - x}$ and $\dfrac{3ax}{4by}$

114. $\dfrac{x^2 - 9x + 20}{x^2 - 6x}$ by $\dfrac{x^2 - 13x + 42}{x^2 - 5x}$

115. $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} - \dfrac{1}{pqr}$ by $pqrs$

116. $\dfrac{x}{3} + \dfrac{x}{12} - \dfrac{x}{5} + \dfrac{x}{20} - \dfrac{x}{4} + \dfrac{x}{15} + \dfrac{x}{30}$ by the L. C. M. of the denominators.

Divide

117. $\dfrac{m}{n}$ by $\dfrac{p}{q}$

118. $\dfrac{1}{m + q}$ by $\dfrac{1}{m - q}$

119. $\dfrac{36a^2b^3x}{25x^3y^2z}$ by $\dfrac{9ab^3x}{5x^2yz}$

120. $4a$ by $\dfrac{4}{5} \cdot \dfrac{a^2}{x}$

121. $\dfrac{12a}{2a - 2}$ by $\dfrac{4a}{a - 1}$

122. $a^2 + \dfrac{1}{a^2} + 2$ by $a + \dfrac{1}{a}$

123. $b + \frac{1}{2}$ by $\frac{2}{3}b + \frac{1}{4}$

124. $\dfrac{a^2 + b^2}{a^2 - b^2}$ by $\dfrac{a - b}{a + b}$

125. a by its reciprocal

126. $\sqrt{x}$ „

127. $a - x$ „

128. $\dfrac{1}{a}$ „

129. $\dfrac{a - b}{c}$ „

130. $x^4 - \dfrac{1}{x^4}$ „

131. $9a^2 - 28 + \dfrac{4}{a^2}$ by $3a - 4 - \dfrac{2}{a}$

132. $\dfrac{x^2 + 3x + 2}{x^2 + 2x + 1}$ by $\dfrac{x^2 + 7x + 12}{x^2 + 5x + 4}$

133. $x^6 + \dfrac{1}{x^6} + x^4 + \dfrac{1}{x^4} + x^2 + \dfrac{1}{x^2} + 2$ by $x^3 + \dfrac{1}{x^3} + x + \dfrac{1}{x}$

134. The sum of $\dfrac{\frac{a}{b} + 1}{\frac{a}{b} - 1}$ and $\dfrac{\frac{a}{b} - 1}{\frac{a}{b} + 1}$ by their difference.

Simplify

135. $\dfrac{1}{1+\dfrac{1}{x}}$

136. $\dfrac{a}{b+\dfrac{c}{d}}$

137. $\dfrac{\dfrac{1}{1+x}}{1-\dfrac{1}{1+x}}$

138. $\dfrac{1}{\dfrac{1}{x-4}-\dfrac{1}{x-3}}$

139. $\dfrac{1+\dfrac{1}{a}}{1+\dfrac{1}{a^2}}$

140. $\dfrac{x^2-\dfrac{11a^2x}{18}-\dfrac{a^2}{9}}{x^2-\dfrac{ax}{3}-\dfrac{2a^2}{3}}$

141. $\dfrac{1}{1+\dfrac{1}{x+\dfrac{1}{x}}}$

142. $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}$

143. $\dfrac{1}{a-1+\dfrac{1}{1+\dfrac{a}{4-a}}}$

144. $\dfrac{x}{1+\dfrac{x}{1+x+\dfrac{x}{1+x+x^2}}}$

145. $\dfrac{a+\dfrac{b-a}{1+ba}}{1-a\dfrac{b-a}{1+ba}}$

146. $\dfrac{\dfrac{1}{1+x}+\dfrac{x}{1-x}}{\dfrac{1}{1-x}-\dfrac{x}{1+x}}$

147. $\dfrac{\dfrac{a}{a+b}+\dfrac{b}{a-b}}{\dfrac{a}{a-b}-\dfrac{b}{a+b}}$

148. $\dfrac{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}$

149. $\dfrac{x}{x-a}-\dfrac{x}{x+a}-\dfrac{\dfrac{x+a}{x-a}-\dfrac{x-a}{x+a}}{\dfrac{x+a}{x-a}+\dfrac{x-a}{x+a}}$

150. $$\frac{3abc}{bc+ca-ab}-\frac{\dfrac{a-1}{a}+\dfrac{b-1}{b}+\dfrac{c-1}{c}}{\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}}$$

RESOLUTION OF FRACTIONS INTO INFINITE SERIES.

81. It has been already remarked, that a fraction indicates a division which cannot be actually performed; but, when an algebraic fraction has a compound denominator, whether the numerator be simple or compound, an attempt to effect the division indicated will always give rise, not to a complete quotient, or one that can be expressed in a finite number of terms, but to a series of terms which may be continued without limit. Thus the fraction

$$\frac{x}{a+b},$$

by actual division of the numerator by the denominator, will give an infinite series for quotient:

$$\begin{array}{l}
a+b\,\big)\,x\,\Big(\dfrac{x}{a}-\dfrac{bx}{a^2}+\dfrac{b^2x}{a^3}-\dfrac{b^3x}{a^4}\ \text{\&c., or} \\
\quad x+\dfrac{bx}{a} \\
\hline
\quad -\dfrac{bx}{a} \\
\quad -\dfrac{bx}{a}-\dfrac{b^2x}{a^2} \\
\hline
\qquad \dfrac{b^2x}{a^2} \\
\qquad \dfrac{b^2x}{a^2}+\dfrac{b^3x}{a^3} \\
\hline
\qquad -\dfrac{b^3x}{a^3} \\
\qquad -\dfrac{b^3x}{a^3}-\dfrac{b^4x}{a^4} \\
\hline
\qquad\quad \dfrac{b^4x}{a^4}\ \text{\&c.}
\end{array}$$

$$\frac{x}{a}\left(1-\frac{b}{a}+\frac{b^2}{a^2}-\frac{b^3}{a^3}\ \text{\&c.,}\right)$$

The remainder will never disappear, and consequently a complete quotient can never be found. The law which the quotient follows, is, however, obvious; each successive term being found by multiplying the one before it by $\frac{b}{a}$, and the signs being alternately *plus* and *minus*

The process may be verified in the usual manner, by multiplying the divisor by any part of the quotient, and adding in the remainder at that part. Thus, in the above example, by multiplying the first four terms by $a + b$, and adding in the remainder $\frac{b^4x}{a^4}$, the dividend x will be reproduced. And the same would be found by taking any other number of terms, and adding the corresponding remainder.

2. Divide $x + a$ by $x + b$.

Here, the quotient is readily found to be,

$$1 + \frac{a-b}{x} - \frac{(a-b)b}{x^2} + \frac{(a-b)b^2}{x^3} -, \text{ \&c.}$$

3. Reduce the fraction $\frac{1}{1+x}$ into an infinite series by division.

Here, proceeding as in the other examples, we find for quotient, $1 - x + x^2 - x^3$, in which the even terms, containing the odd powers of x, are negative, and the others positive. The index of x in any term is one less than the number of terms, x^2 in the 3d term, x^3 in the 4th term, &c.; and the remainder after any term is the next highest power of x. Thus, the hundredth term with remainder will be $-x^{99} + \frac{x^{100}}{1+x}$; the hundred and first with remainder $+x^{100} - \frac{x^{101}}{1+x}$; and the *n*th, or last term, $\pm x^{n-1} \mp \frac{x^n}{1+x}$. This *n*th term is also called the *general* term of the series, because it will give any term if particular values be taken for n. Thus, putting $n = 10$, we have for the tenth term and remainder, $-x^9 + \frac{x^{10}}{1+x}$; and if $n = 15$, we have for the fifteenth term, $+x^{14} - \frac{x^{15}}{1+x}$ and so on. We have, then, for the complete series,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \quad . \quad . \quad . \quad \pm x^{n-1} \mp \frac{x^n}{1+x}$$

But if the sign of x in the divisor be changed, all the terms will be positive, and the series will then be,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \quad . \quad . \quad . \quad + x^{n-1} + \frac{x^n}{1-x}$$

4. If the terms of the divisor, in the two last examples be reversed, the resulting series will be

$$\pm\left(\frac{1}{x} \mp \frac{1}{x^2} + \frac{1}{x^3} \mp \frac{1}{x^4} + . \quad . \quad . \quad . \quad . \quad . \mp \frac{1}{x^n}\right)$$

82. In example 3, last Art. let $x = \frac{1}{3}$; then

$$\frac{1}{1+x} = \frac{1}{1+\frac{1}{3}} = \frac{1}{\frac{4}{3}} = \frac{3}{4} = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \frac{1}{243} \text{ \&c.}$$

If we consider only two terms of this series, the sum will be $\frac{2}{3}$, which is less than the true sum by $\frac{1}{12}$; three terms make $\frac{7}{9}$, which differs from the true sum by $\frac{1}{36}$; four terms make $\frac{20}{27}$ which differs only by $\frac{1}{108}$: the successive differences for five, six, seven, and eight terms will be $\frac{1}{324}$, $\frac{1}{972}$, $\frac{1}{2916}$ and $\frac{1}{8748}$, respectively. Hence we may infer that, if the series were continued to infinity, the difference would disappear, or become less than any assignable quantity, and the sum would be exactly $\frac{3}{4}$.

The fraction $\frac{3}{4}$ is thus the limit towards which the sum continually approaches, but which it cannot pass.

In the series of the first example, let $a = 6$, $b = 4$, $x = 3$; then

$$\frac{x}{a+b} = \frac{3}{10} = \frac{1}{2} - \frac{1}{3} + \frac{2}{9} - \frac{4}{27} + \frac{8}{81}, \text{ \&c.}$$

And, if the first two terms be taken, the sum is too small by $\frac{2}{15}$; if three, it is too large by $\frac{4}{45}$; if four, it is too small by $\frac{8}{135}$; if five, it is too great by $\frac{16}{405}$; and so on. At last, therefore, the sum will be $\frac{3}{10}$; $\frac{3}{10}$ is therefore the limit towards which the sum continually approaches.

83. Similar series sometimes arise in Division: thus, if the terms of the dividend of the first example in Art. 44 be arranged as under, and the arrangement be adhered to, the quotient will be an infinite series. *See Note*, p. 40.

$$a + x)2ax + x^2 + a^2(2x - \frac{x^2}{a} + \frac{x^3}{a^2} - \frac{x^4}{a^3}, \text{ \&c.}$$

The remainders in the third case of Division may also be expanded into series of the same kind. Thus, in example 2, $\frac{2a^4}{x+a}$ may be so expanded, giving for quotient,

$$2(\frac{a^4}{x} - \frac{a^5}{x^2} + \frac{a^6}{x^3} - \frac{a^7}{x^4}, \text{ \&c.})$$

or

$$2\frac{a^4}{x}(1 - \frac{a}{x} + \frac{a^2}{x^2} - \frac{a^3}{x^3} + \frac{a^4}{x^4}, \text{ \&c.})$$

The quotient is therefore interminable, whenever there is such a remainder. *See page* 40.

EXERCISES.

Divide

1. 4 by $1 - a$.
2. a by $a + x$.
3. a by $1 + x$.
4. a by $x + 1$.
5. $a - x$ by $b - x$.
6. $x + 2$ by $x + 3$.
7. 1 by $x^2 - 2x + 1$.
8. $1 - 3x - 2x^2$ by $1 - 4x$.
9. 1 by $1 - 2x + x^2$.
10. What will $\frac{1}{x}$ become, when x is changed into $x + h$?

VIII.—INVOLUTION.

ART. 84. Since (Art. 41) $\frac{a^n}{a^n} = a^{n-n} = a^0$, and also $\frac{a^n}{a^n} = 1$, it follows that $a^0 = 1$. *Hence, that power of any quantity, whose index is zero, is equal to unity.*

Again, since $a^0 = 1$, if we divide each of these equals by a^n, we obtain (Art. 30)

$$a^{0-n}, \text{ or } a^{-n} = \frac{1}{a^n}.$$

Hence, any power with a negative index, is equal to the reciprocal of the same power with a positive index.

85. It follows from Art. 84, 29, 80, that

$$\frac{b}{a^n x} = \frac{b}{x} \cdot \frac{1}{a^n} = \frac{b}{x} \cdot a^{-n} = \frac{ba^{-n}}{x},$$

and

$$\frac{a^n b}{x} = \frac{b}{x} a^n = \frac{b}{x} \div \frac{1}{a^n} = \frac{b}{x} \div a^{-n} = \frac{b}{a^{-n} x};$$

that is to say, that a quantity may be taken from the denominator into the numerator of a fraction, and from the numerator into the denominator, by changing the sign of its index.

86. The square root may be expressed as a power, by using an index, which, when added to itself, will give 1: this index must be $\frac{1}{2}$; and then $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$: hence $a^{\frac{1}{2}} = \sqrt{a}$. In the same manner, since $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a = a^1$, and $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$, we may represent the cube root by the index $\frac{1}{3}$, then $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a$. And, in general, $\sqrt[n]{a} = a^{\frac{1}{n}}$, for ${}^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}}$ &c. to n terms $= a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n} \text{ \&c., to } n \text{ terms}} = a^{\frac{n}{n}} = a^1 = a$. Again, since a quantity is squared, when its index is doubled, and since $\sqrt[3]{a} = a^{\frac{1}{3}}$, it follows that $\sqrt[3]{a^2} = a^{\frac{2}{3}}$; so that $a^{\frac{2}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = a^{\frac{6}{3}} = a^2$. In like manner, $\sqrt[4]{a^3} = a^{\frac{3}{4}}$; for $a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} = a^{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}} = a^{\frac{12}{4}} = a^3$. And, in general, $\sqrt[n]{a^m} = a^{\frac{m}{n}}$, for $a^{\frac{m}{n}} \times a^{\frac{m}{n}}$ &c. to n terms $= a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n} \text{ \&c., to } n \text{ terms}} = a^{\frac{nm}{n}} = a^m$; the nth root of a^m being such, that it produces a^m when repeated n times as a factor.

87. To raise a quantity to any power; *Multiply the quantity continually by itself, until it has been used as a factor as many times as there are units in the index of the required power.*

The reason of the rule is manifest from the nature of powers (Art. 14), involution being merely a succession of multiplications.

EXAMPLES

1. Raise $-a^3b^2x^4$ to the third power.

By continual multiplication, we have

$$-a^3b^2x^4 \times -a^3b^2x^4 \times -a^3b^2x^4 = -a^9b^6x^{12}.$$

2. Involve $4a^{\frac{1}{2}}b^{\frac{2}{3}}$ to the second power.
Here, $4a^{\frac{1}{2}}b^{\frac{2}{3}} \times 4a^{\frac{1}{2}}b^{\frac{2}{3}} = 16ab^{\frac{4}{3}} = 16a\sqrt[3]{b^4}$.

3. Raise $\frac{3a^3}{4x^{\frac{1}{2}}}$ to the fourth power.

Multiplying continually, we have

$$\frac{3a^3}{4x^{\frac{1}{2}}} \times \frac{3a^3}{4x^{\frac{1}{2}}} \times \frac{3a^3}{4x^{\frac{1}{2}}} \times \frac{3a^3}{4x^{\frac{1}{2}}} = \frac{81a^{12}}{256x^2}.$$

4. Involve $a + b$ to the third power.

Multiplying $a + b$ by itself, and the product $a^2 + 2ab + b^2$ again by $a + b$, we obtain

$$a^3 + 3a^2b + 3ab^2 + b^3$$

for the required power. This may be put under the form $a^3 + b^3 + 3ab(a + b)$.

88. Hence, a quantity may be raised to any power, by multiplying its index by the index of the power; that is, $(a^m)^n = a^{mn}$.

For the continued product of a^m into itself n times, is found by adding the index m (Art. 35) as often as there are units in n; that is, multiplying m by n.

89. Hence, also, any power of a product is equal to the product of the same powers of its factors, that is, $(ab)^n = a^nb^n$; for $(ab)^n = ab \times ab \times ab$, &c. to n terms $= a \times a \times a$, &c. to n terms $\times b \times b \times b$, &c. to n terms (Art. 11) $= a^nb^n$.

90. Hence also it follows that, when the quantity to be involved is negative, all the *even* powers will be positive, and all the *odd* powers negative.

91. According to the preceding rule, the successive

powers of the binomial $a+b$, found by continual multiplication, will be as follows :—

$(a+b)^2 = a^2 + 2ab + b^2.$
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$
$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$
$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$
$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6$
$+ b^7.$

By comparing these developments it appears that,

1. The index of a decreases, and that of b increases, by a unit in each successive term; so that the sum of the indices in each term is always equal to the index of the given power. Hence, in general, the terms of $(a+b)^n$, without the coefficients, will be

$$a^n,\ a^{n-1}b,\ a^{n-2}b^2,\ a^{n-3}b^3,\ a^{n-4}b^4,\ a^{n-5}b^5,\ \&c.\ .\ .\ ab^{n-1},\ b^n.^*$$

2. The number of terms is one greater than the index; for there must be as many powers of a as there are units in the index, and one term besides, which does not contain a; so that when the index is even, the number of terms is odd, and *vice versa.*

3. The coefficients increase as far as the middle term, and then diminish, the terms equidistant from it having the same coefficient; thus those of the first and last are unity, of the second and last but one, the same as the given index, &c. And in general, if we know the coefficient in any term, that of the next following will be found by multiplying this coefficient by the index of a in that term, and dividing the product by the number of terms. Thus, in the case of $(a+b)^6$, the coefficient of the second term is 6, the index of

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* The first and last terms may (Art. 84) be written a^nb^0, and a^0b^n.

a in that term is 5, the number of terms is 2; and $\frac{6 \times 5}{2} = 15$, the coefficient of the third term. In like manner, $\frac{15 \times 4}{3} = 20$, the coefficient of the fourth term; and on examining the coefficients of the other powers, we shall find that they may all be formed in the same manner.

According to this law, the coefficients of the terms of $(a+b)^n$ will be as follows:—

The coefficient of the first term 1

The coefficient of the second term .... $\frac{1 \times n}{1} = n.$

The coefficient of the third term .... $\frac{n(n-1)}{2}.$

The coefficient of the fourth term

$$\frac{n(n-1)}{2} \times \frac{n-2}{3} = \frac{n(n-1)(n-2)}{2 \,.\, 3}.$$

The coefficient of the fifth term

$$\frac{n(n-1)(n-2)}{2 \,.\, 3} \times \frac{n-3}{4} = \frac{n(n-1)(n-2)(n-3)}{2 \,.\, 3 \,.\, 4}.$$

&c. &c.

Combining these with the terms of $(a+b)^n$, already given, we have

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2 \,.\, 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{2 \,.\, 3 \,.\, 4}a^{n-4}b^4 + \frac{n(n-1)(n-2)(n-3)(n-4)}{2 \,.\, 3 \,.\, 4 \,.\, 5}a^{n-5}b^5 + \&c. \;\ldots\ldots + nab^{n-1} + b^n.$$

This is the celebrated BINOMIAL THEOREM, by means of which a binomial quantity may be involved to any power without the trouble of continual multiplication.*

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* A general demonstration of the binomial theorem will be given farther on. It is convenient to introduce the theorem here for practical

92. If the second term, b, of the binomial be negative, its even powers will be positive, and its odd powers negative (Art. 90); but the even terms of the expanded power contain the odd powers of b, and therefore the powers of a being all positive, the odd terms of the powers of $a - b$ will be positive, and the even terms negative. In other respects, the powers of $(a-b)^n$ will be the same as those of $(a+b)^n$.

Further, it is plain from the manner in which the developments are generated, that the series will terminate at the $(n+1)th$ term, when n is a whole positive number, since, on passing that term, $n-n$ or zero would always occur as a factor, and therefore every higher term would vanish. But assuming that the series holds when n is negative or fractional, n will not, in this case, be destroyed by any of the numbers successively taken from it. Thus no factor of the coefficients can become zero, and therefore the series will never terminate.

EXAMPLES.

1. Required the 6th power of $2x^2 - 3y$.

In this case we must put $2x^2$, $-3y$, and 6, for a, b, and n, in the general formula. Hence,

The first term is .... $(2x^2)^6 = 64x^{12}$.
The second..................... $6 \times (2x^2)^5 \times (-3y) = -576x^{10}y$.
The third .... $6 \times \frac{5}{2} \times (2x^2)^4 \times (-3y)^2 = 2160x^8y^2$.
The fourth .... $15 \times \frac{4}{3} \times (2x^2)^3 \times (-3y)^3 = -4320x^6y^3$.
The fifth .... $20 \times \frac{3}{4} \times (2x^2)^2 \times (-3y)^4 = 4860x^4y^4$.
The sixth .... $15 \times \frac{2}{5} \times 2x^2 \times (-3y)^5 = -2916x^2y^5$.
The seventh .... $6 \times \frac{1}{6} \times (2x^2)^0 \times (-3y)^6 = 729y^6$.

---

purposes; and the above inductive proof is the less objectionable, as no subsequent demonstration is founded upon it, and as it seems to have been by a similar process that Newton himself inferred the generality of this important formula.

Combining these, we have

$$(2x^2 - 3y)^6 = 64x^{12} - 576x^{10}y + 2160x^8y^2 - 4320x^6y^3$$
$$+ 4860x^4y^4 - 2916x^2y^5 + 729y^6.$$

2. It is required to expand $\frac{x}{(a+b)^2}$ by the binomial theorem.

Since (Art. 85) $\frac{x}{(a+b)^2} = x(a+b)^{-2}$, we shall first expand $(a+b)^{-2}$, and then multiply all the terms by x. Here $n = -2$, therefore

The first term is .... a^{-2}.

The second .... $-2a^{-3}b$.

The third .... $-2 \times \frac{-3}{2} a^{-4}b^2 = 3a^{-4}b^2$.

The fourth .... $3 \times \frac{-4}{3} a^{-5}b^3 = -4a^{-5}b^3$.

The fifth .... $-4 \times \frac{-5}{4} a^{-6}b^4 = 5a^{-6}b^4$.

&c. &c.

That the series will never terminate may also be shewn by actual division; for (Art. 81) $(a+b)^{-2}$, or $\frac{1}{(a+b)^2}$ may be expanded into an infinite series.

We have therefore

$$(a+b)^{-2} = a^{-2} - 2a^{-3}b + 3a^{-4}b^2 - 4a^{-5}b^3 + 5a^{-6}b^4 \text{ \&c.}$$

or (Art. 85)

$$\frac{1}{(a+b)^2} = \frac{1}{a^2} - \frac{2b}{a^3} + \frac{3b^2}{a^4} - \frac{4b^3}{a^5} + \frac{5b^4}{a^6} \text{ \&c.}$$

Hence,

$$\frac{x}{(a+b)^2} = \frac{x}{a^2}\left(1 - \frac{2b}{a} + \frac{3b^2}{a^2} - \frac{4b^3}{a^3} + \frac{5b^4}{a^4} \text{ \&c.}\right)$$

Precisely the same series would be found by dividing x by $a^2 + 2ab + b^2$ (Art. 81).

3. Raise $a + b + c$ to the third power.

Regarding $a + b$ as a single term, we may express $(a + b + c)^3$ under the form $\{(a + b) + c\}^3$; then

The first term is .... $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.
The second .... $3(a + b)^2c = 3a^2c + 6abc + 3b^2c$.
The third .... $\frac{3 \times 2}{2}(a + b)c^2 = 3ac^2 + 3bc^2$.
The fourth .... $\frac{3 \times 1}{3}c^3 = c^3$.

Therefore,

$[(a + b) + c]^3 = (a + b + c)^3 = a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3 = (a + b)^3 + 3(a + b)^2c + 3(a + b)c^2 + c^3 = a^3 + b^3 + c^3 + 3a^2(b + c) + 3b^2(a + c) + 3c^2(a + b) + 6abc$.

The last but one of these forms is the same as that of the cube of a binomial; the last is the most readily applicable in finding the cube of any trinomial. The cubes of polynomials have the same form, the products depending on the various combinations of which the letters entering into them are susceptible.

From the form of the square of a trinomial (Art. 38, *Note*), $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$,
we may at once derive that of any polynomial; it will consist of *the square of each term, together with twice the product of every pair of terms*; that is,
$(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$.

A different arrangement of the products, thus,
$(a + b + c + d)^2 = a^2 + 2a(b + c + d) + b^2 + 2b(c + d) + c^2 + 2cd + d^2$
shews that the square may also be said to consist of *the square of each term, together with twice the product of each term by the sum of all the terms which follow it.*

EXERCISES.

1. Multiply $3a^{-4}$ by $5a^{6}$.
2. Divide a^{3} by a^{-4}.
3. Express $6a^{-2} + b^{2}x^{-3}$ with positive indices.
4. Express $\frac{bx^{2}}{a^{3}y^{-2}}$ without denominator.
5. Divide $12b^{4}x^{-2}$ by $4b^{-3}x^{4}$.
6. Express $ax^{-(n-1)}$ as a fraction.
7. Express $\frac{b}{a^{4-n}}$ without denominator.
8. Divide $x^{\frac{2}{3}}$ by x.
9. Express $\frac{b}{x\sqrt[4]{x}}$ without denominator.
10. Divide $x^{-\frac{4}{3}}$ by x.
11. Multiply $\frac{1}{2}\,a^{-n}$ by a^{n}.
12. Raise $-2a^{3}b^{\frac{1}{2}}$ to the 4th power.
13. Raise $\sqrt[5]{a^{2}}$ to the 6th power.
14. Raise $\sqrt[3]{(a+x)}$ to the 6th power.
15. Raise $4a^{5}b$ to the nth power.
16. Express $\frac{1}{2}(1-x)(1+x^{2})^{-\frac{1}{2}} \times 2x$ with a positive index.
17. Raise $a-6$ to the 6th power.
18. Raise $5+4x$ to the 4th power.
19. Raise $1+x$ to the nth power.
20. Expand $\frac{x}{(a+b)}$ by the binomial theorem.
21. Expand $\frac{1}{(a+2b)^{3}}$ by the binomial theorem.
22. Find the square of the sum of x and its reciprocal.
23. Express $x^{-\frac{2}{3}}$ with a positive index.
24. Raise $x^{3}+3y^{2}$ to the 5th power.

25. What are 10^{-2}, 2^{-5}, $\frac{1}{7^{-2}}$ and $\frac{3}{5^{-3}}$?

26. Expand $\frac{2}{(a+x)^2}$.

27. Raise $1+x$ to the power whose index is $n+1$.

28. Required the 5th term of $(a^2-x^2)^{12}$.

29. Required the middle term of $(a-x)^{16}$.

30. Required the coefficient of x^5 in the expansion of $(x+a)^8$.

31. Expand $(1-x)^{-n}$.

32. Required the 10th power of $a-1$.

33. Expand $\frac{1}{(1-x)^3}$.

34. Expand $\frac{1}{(1+x^2)^3}$.

35. Divide $\sqrt{x}$ by x.

36. Raise x^3 to the 4th power.

37. „ $\sqrt{x}$ to the 4th power.

38. „ $\frac{1}{a^5}$ to the 5th power.

39. „ $5\sqrt{a}$ to the 3d power.

40. „ $\frac{2}{3}a^n$ to the 4th power.

41. „ $\sqrt[4]{a^3}$ to the 4th power.

42. „ $4a^{-2}b^3$ to the 3d power.

43. „ $-\frac{1}{2}a^2$ to the 2d power.

44. „ $-a^2x^3$ to the 5th power.

45. „ $\frac{1}{3}\sqrt[3]{a^2}$ to the 3d power.

46. „ x^a to the bth power.

47. „ $a^m b^n c^p$ to the rth power.

48. „ $\sqrt[4]{x}$ to the 2d power.

49. Express $\frac{3}{2}a(1-x)^{-\frac{1}{2}}$ with a positive index.

50. „ $a^2x^{-2}-ab^2x^{-3}+bx^{-1}+a^{-m-n}$, with positive indices.

51. Express $\frac{a^4}{x^3} + \frac{2a^3}{x^2} - \frac{ab}{x} + \frac{1}{a^{n+q}}$ without denominators.

52. Find the cube of $\sqrt[n]{a^m}$.

53. „ 6th power of $3x + 2y$.

54. Express $\frac{1}{64}$, $\frac{1}{225}$, $\frac{1}{625}$, and ·001 as powers with negative indices.

55. Square $x^{-1} - y^{-1}$, both in this form, and with positive indices.

56. Raise $\sqrt[4]{(a+b)}$ to the 5th power.

57. „ $\frac{1}{x}$ to the nth power.

58. Square $3a^2 + 2bc + 4x^3$ by inspection.

59. Find the square $1 + x + x^2 + x^3$ in the same manner.

60. Raise $x^2 - 2x + 1$ to the 3d power; and $1 - 3x + 3x^2 - x^3$ to the 2d power, in the same manner.

IX. EVOLUTION.

93. To extract any root of a simple quantity, *Find what is the quantity which, when raised to the corresponding power, will produce the given quantity.*

The reason of the rule is manifest from the nature of roots (Art. 15).

EXAMPLES.

1. Required the third root of $64a^6b^{12}$.

Here it is required to find of what quantity $64a^6b^{12}$ is the third power: the quantity is plainly $4a^2b^4$, because $(4a^2b^4)^3 = 64a^6b^{12}$ (Art. 87); therefore $\sqrt[3]{(64a^6b^{12})} = 4a^2b^4$.

2. Find the square root of $a^{-6}b^{\frac{1}{2}}x^{3}$.

Since $(a^{-3}b^{\frac{1}{4}}\sqrt{x^{3}})^{2} = a^{-6}b^{\frac{1}{2}}x^{3}$, the required root is $a^{-3}b^{\frac{1}{4}}\sqrt{x^{3}} = \dfrac{b^{\frac{1}{4}}x^{\frac{3}{2}}}{a^{3}}$.

3. Find the cube root of $-27a^{6}b^{-9}x^{2}$.

Here $(-3a^{2}b^{-3}x^{\frac{2}{3}}) \times (-3a^{2}b^{-3}x^{\frac{2}{3}}) \times (-3a^{2}b^{-3}x^{\frac{2}{3}}) = -27a^{6}b^{-9}x^{2}$; and the required root is therefore $-3a^{2}b^{-3}x^{\frac{2}{3}}$.

4. Find the fourth root of the cube root of x.

Here, since $\sqrt[3]{x} = x^{\frac{1}{3}}$, the index of the required root must be such that, when it is added four times, it will produce $\frac{1}{3}$: it must be $\frac{1}{12}$, therefore $\sqrt[4]{(\sqrt[3]{x})} = (x^{\frac{1}{3}})^{\frac{1}{4}} = x^{\frac{1}{12}} = \sqrt[12]{x}$.

So also $\sqrt[n]{\sqrt[m]{a}}$ or $(a^{\frac{1}{m}})^{\frac{1}{n}} = a^{\frac{1}{mn}}$, because $a^{\frac{1}{mn}}$ involved to the nth power, gives $a^{\frac{n}{mn}} = a^{\frac{1}{m}}$. . . . Similarly $(a^{\frac{m}{n}})^{\frac{p}{q}} = a^{\frac{mp}{nq}}$; for $(a^{\frac{m}{n}})^{\frac{p}{q}} = (a^{\frac{mp}{n}})^{\frac{1}{q}} = a^{\frac{mp}{nq}}$.

94. Hence, any root of a given power is extracted by dividing the index of the power by the index of the root, or $\sqrt[n]{a^{m}} = (a^{m})^{\frac{1}{n}} = a^{\frac{m}{n}}$; for $a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}}$ &c., to n terms $= a^{\frac{mn}{n}} = a^{m}$.*

95. Hence also, any root of a product is equal to the product of the same root of each of its factors, that is, $\sqrt[n]{(ab)} = \sqrt[n]{a} \times \sqrt[n]{b}$, because each of these, by involution, produces the quantity ab.

96. Hence, and from Art. 90, it follows, that the even roots of all positive quantities may be either positive or negative: and that the odd roots of positive quantities are

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* From this principle we might have derived the method of representing roots as powers with fractional indices (Art. 86); for $\sqrt[n]{a^{m}} = a^{\frac{m}{n}}$, the denominator expressing the root, and the numerator the power whose root is taken.

positive, and those of negative quantities negative; that is to say, $\sqrt[2n]{+a} = \pm\sqrt[2n]{a}$; $\sqrt[2n+1]{+a} = +\sqrt[2n+1]{a}$; and $\sqrt[2n+1]{(-a)} = -\sqrt[2n+1]{a}$. But an even root of a negative quantity is not assignable by any process of evolution: for we know that $(+a) \times (+a) = +a^2$, and that $(-a) \times (-a) = +a^2$, and, therefore, that $\sqrt{a^2} = \pm a$; but we know of no quantity which, multiplied by itself, will produce $-a^2$: and the same must hold whenever the index of the root is even: therefore, $\sqrt[2n]{(-a)}$ is not assignable.

EXERCISES.

Extract the roots indicated by the several signs.

1. $\sqrt[3]{(-8a^6b^{15})}$
2. $\sqrt{(9x^2b^4)}$
3. $\sqrt{(x^{2n})}$
4. $\sqrt[n]{(x^{2n})}$
5. $(16a^{16})^{\frac{1}{4}}$
6. $(x^{3})^{\frac{1}{3}}$
7. $\sqrt[3]{(-125a^3b^6)}$
8. $\sqrt{\dfrac{4a^6}{9b^4}}$
9. $(x^{-\frac{1}{3}})^{-\frac{1}{2}}$
10. $\sqrt{(\sqrt{y})}$
11. $\left(\dfrac{32x^{10}y^5}{243b^{20}a^{15}}\right)^{\frac{1}{5}}$
12. $\sqrt[5]{x^{5}}$
13. $\sqrt[m]{(a^{-mn})}$
14. $\sqrt{(5a^4b^{-6}x^{4n})}$
15. $(-27a^2b^{\frac{1}{3}}x^{-8})^{\frac{1}{3}}$
16. $\left(\dfrac{a^{-1}}{b^{3n}x^{\frac{1}{n}}}\right)^{\frac{1}{3}}$
17. $\sqrt{\sqrt[3]{\sqrt[4]{a}}}$
18. $\left(\dfrac{3^{10}a^{-5}(a+b)^5(2+x)^{-10}}{32c^5d^{-15}}\right)^{\frac{1}{5}}$
19. $[(a^{-1})^{-2}]^{-\frac{1}{2}}$
20. $\sqrt[3]{\dfrac{27x^{12}(a+x)^{-6n}}{8b^{-3n}c^9}}$
21. $(x^{16})^{\frac{1}{4}}$
22. $(x^{30})^{\frac{1}{10}}$
23. $(x^{2n+4})^{\frac{1}{2}}$
24. $\sqrt{(a+x)^6}$
25. $(9a^4x^2y^{-12})^{\frac{1}{2}}$
26. $(169x^{2m+2}a^{-2n})^{\frac{1}{2}}$
27. $\sqrt[3]{(a^2+x^2)^{\frac{1}{2}}}$
28. $\sqrt[3]{\dfrac{64a^9x^{12}y^{-15}}{125x^6(a+x)^3}}$
29. $\sqrt[3]{-\dfrac{343a^{12}x^{-3}y^4}{729b^9c^{15}z}}$
30. $\left(\dfrac{32a^{10}y^{15}}{6^5}\right)^{\frac{1}{5}}$
31. $\left[\overline{(x^{\frac{1}{2}})^{\frac{1}{3}}}^{\frac{1}{6}}\right]^{\frac{1}{2}}$
32. $\left(\dfrac{1024b^{100}x^5z^{10}}{10{,}000{,}000{,}000a^{30}}\right)^{\frac{1}{10}}$
33. $\sqrt[3]{[(a^{\frac{1}{5}})^{\frac{1}{2}}]^{\frac{1}{3}}}$

COMPOUND QUANTITIES.

97. From what has been already stated regarding the nature of evolution, it is plain that the roots of compound quantities will be found by reversing the process of involution. The law according to which powers are generated being known, we are enabled to construct the steps of a converse process, by which the terms of the root are evolved one by one from the given power. By successive divisions the given quantity is diminished by the greatest power contained in it, of the same order as the root which is to be extracted. Each division gives a new term of the root; so that we take away in succession the greatest power of a monomial (of this order), the greatest of a binomial, the greatest of a trinomial, and so on; until at last there is either no remainder—in which case all the terms of the given quantity are exhausted, and it is of course seen to be a complete power—or else there is a remainder, which by a continuation of the process will lead, as in Division, to an interminable quotient (Arts. 44, 81). It is obvious that the terms of the given quantity must be arranged, as in Division, according to the powers of some one letter.

The forms assumed by the squares and cubes of binomials, trinomials, &c., have been already given, and the law of their formation stated (Arts. 38, 91, 92, Ex. 3.) From these we shall now derive rules for the extraction of the square root and cube root.

Let it be required to find the square root of $a^2 + 2ab + b^2$.

This and the following examples are selected, although we know the roots by inspection, because they give the general forms of the square of a binomial, and of a trinomial, and place in a clear light the reason of the converse process.

The first term of the root is plainly a; subtracting the square of this, there remains $2ab + b^2$. Now since $2ab$, the second term of the square, is twice the product of the two terms of the root (Art. 38, Note), that is, the product of the last term of the root and twice the first, if this second term $2ab$ be divided by $2a$, twice the first, the quotient will be the second term, b, of the root. Then, in order to find whether the two last terms of the given quantity will be exactly reproduced, the second part, b, of the root must be joined to the *trial* divisor $2a$, giving $2a + b$ for the *complete* divisor.

$$\begin{array}{r|l}
 & \underline{a + b} \\
 & a^2 + 2ab + b^2 \\
 & \underline{a^2} \\
2a + b & 2ab + b^2 \\
 & \underline{2ab + b^2} \\
 & 0
\end{array}$$

Multiplying this by b, and subtracting, we find that there is no remainder; and $a + b$ is therefore the complete root.

Again, let the quantity whose square root is required, be $a^2 + 2ab + b^2 + 2c(a + b) + c^2$.

Here, the first and second terms of the root are determined as before; we then repeat the same process upon the remainder, dividing by the double of $a + b$, as we before divided by the double of a; we thus find c for the quotient; when it is connected to the root, and to the trial divisor, $2(a + b)$, and the product taken, there is no remainder, and the root is therefore $a + b + c$.

$$\begin{array}{r|l}
 & \underline{a + b + c} \\
 & a^2 + 2ab + b^2 + 2c(a + b) + c^2 \\
 & \underline{a^2 } \\
2(a + b) & 2ab + b^2 \\
 & \underline{2ab + b^2} \\
2(a + b) + c & 2c(a + b) + c^2 \\
 & \underline{2c(a + b) + c^2} \\
 & 0
\end{array}$$

In these examples, the given quantity is diminished, first by a^2, the power of a monomial, corresponding to the degree of the root; the second step being complete, it is diminished in all by $a^2 + 2ab + b^2$, the like power of a binomial; at the close of the third step by the same power of a trinomial, and so on.

The third step in this case exhausts all the terms, but it is obvious that the principle is applicable to all polynomials.

The formation of the trial divisor is the most important part of the process of evolution. An arrangement of its terms, slightly different from the above, will better exhibit its structure and mode of increase, and thus render its formation more easy in all cases.

$$\begin{array}{l|l}
 & \qquad\qquad\qquad \underline{a+b+c} \\
 & a^2 + 2ab + b^2 + 2ac + 2bc + c^2 \\
a & a^2 \\ \hline
\underline{a} & \quad 2ab + b^2 \\
2a & \quad 2ab + b^2 \\ \hline
\underline{\quad +b} & \qquad\qquad 2ac + 2bc + c^2 \\
2a+b & \qquad\qquad \underline{2ac + 2bc + c^2} \\
\underline{\qquad b} & \qquad\qquad\qquad 0 \\
2a+2b & \\
\underline{\qquad\quad +c} & \\
2a+2b+c &
\end{array}$$

The root a of the first term is placed on the same line as the first term a^2, which is taken away; this first term a of the root is then added to itself, to obtain the first trial divisor $2a$, in the same way as b, c, &c. are successively added afterwards to the parts of the root already found. This gives a greater uniformity to the process of forming the divisor. The horizontal lines are connected by vertical bars, merely to refer more distinctly each trial divisor to its own dividend. To exhibit the whole process of formation as a succession of additions, we may place a cipher to the left of the given quantity, heading the column of the trial divisor, and go

$$\begin{array}{ll}
0 & a^2 + 2ab + b^2 + 2ac + 2bc + c^2 \\
\underline{a} & \underline{a^2 \qquad\qquad\qquad} \\
a & \quad 2ab + b^2, \text{ \&c.} \\
\underline{a} & \\
2a & \\
\underline{\quad +b} & \\
2a+b, \text{ \&c.} &
\end{array}$$

through the form of adding a to this, then add to find $2a$ as before, and so on.

The relation of the two processes, the direct and the reverse, will be more clearly seen from the following scheme, in which the successive multiplications and additions, by which the power is made up under a form similar to the preceding, are contrasted with the succession of divisions and subtractions in the exhaustive process for finding the root. The method, which is synthetic, is plainly derivable from an analytical examination of the factors of the component terms of developed powers. Thus we may analyse the square of $a+b$, either into $a \times a + (2a+b)b$, or into $b \times b + (2b+a)a$, and so on, as presented in the annexed scheme.

To find the square of $a+b+c+d$, two columns are headed with cipher; each successive term is added twice in

$$
\begin{array}{ll}
0 & 0 \\
+a & a^2 \\
a & a^2 \\
+a & 2ab+b^2 \\
2a & a^2+2ab+b^2 \\
+b & 2ac+2bc+c^2 \\
2a+b & a^2+2ab+b^2+2ac+2bc+c^2 \\
b & 2ad+2bd+2cd+d^2 \\
2a+2b & a^2+2ab+b^2+2c(a+b)+c^2 \\
+c & +2d(a+b+c)+d^2 \\
2a+2b+c & \\
c & \\
2a+2b+2c & \\
+d & \\
2a+2b+2c+d & \\
d & \\
2a+2b+2c+2d &
\end{array}
$$

the first column, and before the second addition is performed each sum is multiplied by the quantity added, and the sum placed in the second column, in which it is only added once. Thus before the second addition a is multiplied by a to obtain a^2; $2a + b$ is multiplied by b to find $2ab + b^2$, which is then added in the second column, and so of the others. The spaces marked off by lines refer these products to the quantities from which they arise; $2ab + b^2$ from $2a + b$ and b, and so on. The generation of the terms is thus more distinctly shewn, the only use being to prevent confusion. The squares of polynomials may be very readily formed by this process.

EXAMPLES.

1. Required the square root of $9x^6 + 24x^5 + 4x^4 + 26x^3 + 60x^2 - 28x + 49$.

$$
\begin{array}{ll}
 & \qquad\qquad\qquad\qquad\qquad \overline{3x^3 + 4x^2 - 2x + 7} \\
0 & 9x^6 + 24x^5 + 4x^4 + 26x^3 + 60x^2 - 28x + 49 \\
\underline{3x^3} & \underline{9x^6} \\
3x^3 & \quad 24x^5 + 4x^4 \\
\underline{3x^3} & \quad \underline{24x^5 + 16x^4} \\
6x^3 & \qquad -12x^4 + 26x^3 + 60x^2 \\
\underline{\quad + 4x^2} & \qquad \underline{-12x^4 - 16x^3 + 4x^2} \\
6x^3 + 4x^2 & \qquad\qquad 42x^3 + 56x^2 - 28x + 49 \\
\underline{\qquad 4x^2} & \qquad\qquad \underline{42x^3 + 56x^2 - 28x + 49} \\
6x^3 + 8x^2 & \qquad\qquad\qquad 0 \\
\underline{\qquad\qquad - 2x} & \\
6x^3 + 8x^2 - 2x & \\
\underline{\qquad\qquad - 2x} & \\
6x^3 + 8x^2 - 4x + 7 &
\end{array}
$$

In order to save room, the column containing the successive divisors may be contracted as on the margin. The work may also be much shortened by using the method of detached coefficients.

$$\begin{array}{r} 3x^3 \\ 3x^3 \\ \hline 6x^3+4x^2 \\ 4x^2 \\ \hline 6x^3+8x^2-2x \\ -2x \\ \hline 6x^3+8x^2-4x+7 \end{array}$$

2. Required the square root of $\frac{a^2}{b^2}+\frac{b^2}{a^2}-2$.

Here, since $\frac{b^2}{a^2}=b^2a^{-2}$ and $2=2a^0$, 2 is a higher power of a than $\frac{b^2}{a^2}$ is; the quantity is therefore arranged as under:

$$\begin{array}{lll} \frac{a}{b} & \frac{a^2}{b^2}-2+\frac{b^2}{a^2} & \Big(\frac{a}{b}-\frac{b}{a} \\ \frac{a}{b} & \frac{a^2}{b^2} & \\ \cline{1-2} \frac{2a}{b}-\frac{b}{a} & -2+\frac{b^2}{a^2} & \\ & -2+\frac{b^2}{a^2} & \\ \cline{2-2} & 0 & \end{array}$$

3. Required the square root of $1+x$.

$$\begin{array}{lll} & 1+x & \Big(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\&c. \\ 1 & 1 & \\ \cline{2-2} 1 & +x & \\ \cline{1-1} 2+\frac{x}{2} & +x+\frac{x^2}{4} & \\ \cline{2-2} \frac{x}{2} & -\frac{x^2}{4} & \\ \cline{1-1} 2+x-\frac{x^2}{8}\ \&c. & -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64}\ \&c. & \end{array}$$

The next two steps would give $\frac{x^3}{16} - \frac{5x^4}{128}$.

If the numerators and denominators of the two last terms of this root be multiplied by 3, the series may be put under the form,

$$1 + \frac{x}{2} - \frac{x^2}{2 . 4} + \frac{3x^3}{2 . 4 . 6} - \frac{3 . 5x^4}{2 . 4 . 6 . 8} + \frac{3 . 5 . 7x^5}{2 . 4 . 6 . 8 . 10} + \&c.$$

The law according to which the series is continued is now manifest.

4. Find the value of $\sqrt{(x^2 + ax)}$ in a series.

Here, using detached coefficients, the work will be as follows :

$$\begin{array}{lll}
 & 1 + 1 & \underline{\left| 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} + \frac{7}{256} \right.} - \&c. \\
1 & \underline{1} & \\
\underline{1} & +1 & \\
2 + \frac{1}{2} & \underline{1 + \frac{1}{4}} & \\
\underline{\frac{1}{2}} & -\frac{1}{4} & \\
2 + 1 - \frac{1}{8} & \underline{-\frac{1}{4} - \frac{1}{8} + \frac{1}{64}} & \\
\underline{-\frac{1}{8}} & \phantom{-\frac{1}{4}}+\frac{1}{8} - \frac{1}{64} & \\
2 + 1 - \frac{1}{4} + \frac{1}{16} & \underline{+\frac{1}{8} + \frac{1}{16} - \frac{1}{64} + \frac{1}{256}} & \\
\&c. & \phantom{+\frac{1}{8}}-\frac{5}{64} + \frac{1}{64} - \frac{1}{256},\ \&c. &
\end{array}$$

The next two steps in the process would give for quotient $-\frac{5}{128} + \frac{7}{256}$. If now we supply the letters, the root will be

$$x^1 + \frac{1}{2} . ax^0 - \frac{1}{8} . a^2x^{-1} + \frac{1}{16} . a^3x^{-2} - \frac{5}{128} . a^4x^{-3} + \frac{7}{256} . a^5x^{-4} - \&c.$$

Now, $\frac{1}{8} = \frac{1}{2 . 4}$; $\frac{1}{16} = \frac{1}{2 . 4 . 2} = \frac{1 . 3}{2 . 4 . 6}$; $\frac{5}{128} = \frac{5}{16 . 8}$ $= \frac{1 . 3 . 5}{2 . 4 . 6 . 8}$; $\frac{7}{256} = \frac{7}{128 . 2} = \frac{5 . 7}{128 . 10} = \frac{1 . 3 . 5 . 7}{2 . 4 . 6 . 8 . 10}$.

Hence we have for the required value (Art. 85),

$$\sqrt{(x^2+ax)} = x + \frac{1}{2}a - \frac{1 \cdot a^2}{2 \cdot 4x} + \frac{1 \cdot 3a^3}{2 \cdot 4 \cdot 6x^2} - \frac{1 \cdot 3 \cdot 5a^4}{2 \cdot 4 \cdot 6 \cdot 8x^3}$$
$$+ \frac{1 \cdot 3 \cdot 5 \cdot 7a^5}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10x^4} - \&c.$$

From which the law of the series is obvious.

5. Find on what condition the expression $a^2x^4 + bx^2 + c^2$ shall be a complete square.

Extracting the root by the preceding process, we find $ax^2 + \frac{b}{2a}$ with a remainder $c^2 - \frac{b^2}{4a^2}$. Now, in order that the expression may have a finite square root there must be no remainder, that is, we must have $c^2 = \frac{b^2}{4a^2}$; this, therefore, is the condition, and the expression in the form of a complete square is $a^2x^4 + bx^2 + \frac{b^2}{4a^2}$.

6. Show that $x^4 + px^3 + qx^2 + rx + s$, is a complete square, if $p^2s = r^2$, and $q = \frac{p^2}{4} + 2\sqrt{s}$.

Suppose that it is a perfect square, and that it is represented by the square of the trinomial, $x^2 + ax + b$: then the given expression must be identical with $x^4 + 2ax^3 + (a^2 + 2b)x^2 + 2abx + b^2$; so that we must have $p = 2a$, $q = a^2 + 2b$, $r = 2ab$, and $s = b^2$, whence $p^2s = 4a^2b^2 = r^2$, and $q = a^2 + 2b = \frac{p^2}{4} + 2\sqrt{s}$.

98. The process for the extraction of the cube root is derived from the form of the cube of a binomial, and our knowledge of the mode of generation of the power, as already explained in the case of the second power. The second term

of the cube of a binomial is $3a^2b$; that is, the product of the second term of the root by three times the square of the first term; so that to find b, the second term of the root, $3a^2$, that is, *three times the square of the first part of the root*, must be used as the trial divisor. Also, since the cube of $a+b$ may be put under the form $a^3 + (3a^2 + 3ab + b^2)b$, after a^3 has been taken away, the full divisor, which, with b as multiplier, will complete the exhaustive process, is $3a^2 + 3ab + b^2$; so that to $3a^2$, the trial divisor already found, $3ab + b^2 = (3a + b)b$ must be added.

$$\begin{array}{r|l}
 & \underline{a + b} \\
 & a^3 + 3a^2b + 3ab^2 + b^3 \\
 & \underline{a^3 } \\
3a^2 + 3ab + b^2 & 3a^2b + 3ab^2 + b^3 \\
 & \underline{3a^2b + 3ab^2 + b^3} \\
 & 0
\end{array}$$

The mode of generating the power set forth in the following scheme will shew how these several quantities $3a$, $3a^2$, $3a + b$, &c. are produced by successive additions and multiplications, in the form best adapted for establishing the steps of the reverse process.

Three columns are headed with cipher; in the first there are at each step *three* additions, in the second, *two*, and in the third, *one*; the second column is formed from the first, and the third from the second, by multiplying by the new quantity introduced at each step; after one addition in the first column the sum is multiplied by this new quantity, to find the first term to be added in the second column; and this addition being made, the sum is multiplied by the same quantity to find the term in the third column. Thus, $0 + a = a$; then $a \times a = a^2$; $0 + a^2 = a^2$; then $a^2 \times a = a^3$. Adding again on the first column, and multiplying, we have $2a$ and $2a^2$; a third addition in the first column completes the first step, and the three sums are $3a$, $3a^2$, and a^3. In the same way for the next step, after one addition, the sum $3a + b$

$$
\begin{array}{lll}
0 & 0 & 0 \\
\underline{a} & \underline{a^2} & \underline{a^3} \\
a & a^2 & a^3 \\
\underline{a} & \underline{2a^2} & \underline{3a^2b + 3ab^2 + b^3} \\
2a & 3a^2 & a^3 + 3a^2b + 3ab^2 + b^3 \\
\underline{\quad a} & \underline{\qquad 3ab + b^2} & \\
3a & 3a^2 + 3ab + b^2 & \text{or, } (a + b)^3 \\
\underline{\quad + b} & \underline{\qquad 3ab + 2b^2} & \\
3a + b & 3a^2 + 6ab + 3b^2 & \\
\underline{\qquad b} & \text{or, } 3(a + b)^2 & \\
3a + 2b & & \\
\underline{\qquad\quad b} & & \\
3a + 3b & & \\
\text{or, } 3(a + b) & &
\end{array}
$$

is multiplied by b, giving $3ab + b^2$, the quantity to be added in the second column; and this being added, the sum multiplied by b is put in the third column. The new term b is again added in the first column, then $(3a + 2b)b = 3ab + 2b^2$ is the second term to be added in the second column; a third addition in the first column, and the addition of the terms already in the other columns, complete the second step, giving the three sums $3(a + b)$, $3(a + b)^2$, and $(a + b)^3$. If a third term c were introduced, we should obviously obtain for the result of the third step, $3(a + b + c)$, $3(a + b + c)^2$, and $(a + b + c)^3$.

The converse process for the discovery of the root from the given power is now easily constructed. Let the cube root of $a^3 + b^3 + c^3 + 3a^2(b + c) + 3b^2(a + c) + 3c^2(a + b) + 6abc$ be required. Ordering the terms, and heading two columns with cipher, the process will be as follows :—

$$
\begin{array}{lll}
 & & \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overline{a+b+c} \\
0 & 0 & a^3+3a^2b+3ab^2+b^3+3a^2c+3b^2c+3ac^2+3bc^2+c^3+6abc \\
\underline{a} & \underline{a^2} & \underline{a^3\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad} \\
a & a^2 & \quad 3a^2b+3ab^2+b^3 \\
\underline{a} & \underline{2a^2} & \underline{\quad 3a^2b+3ab^2+b^3\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad} \\
2a & 3a^2 & \quad\quad\quad\quad\quad\quad 3a^2c+3ac^2+c^3+3b^2c+3bc^2+6abc \\
\underline{a} & \underline{\quad 3ab+b^2} & \quad\quad\quad\quad\quad\quad \underline{3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3} \\
3a & 3a^2+3ab+b^2 & \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 0 \\
\underline{+b} & \underline{\quad 3ab+2b^2} & \\
3a+b & 3a^2+6ab+3b^2 & \\
\underline{b} & \underline{\quad\quad\quad\quad\quad 3ac+3bc+c^2} & \\
3a+2b & 3a^2+6ab+3b^2+3ac+3bc+c^2 & \\
\underline{b} & & \\
3a+3b & & \\
\underline{+c} & & \\
3a+3b+c & &
\end{array}
$$

Here the second column, constructed as above described, contains the trial divisors, the first $3a^2$, and the second, formed after two additions, $3a^2 + 6abc + 3b^2$, each being three times the square of the part of the root already found; the *complete* divisors being the quantities found after *one* addition to each of these; the second and third terms of the root are found by dividing the first terms of the two remainders, $3a^2b$ and $3a^2c$ by the first terms of the trial divisors. The most important part of the process, and that in which mistakes are most likely to occur, is the formation of this second column. It should be carefully observed that the two products successively added in it are found by multiplying the sum in the first column by the new term of the root, first, after the first addition in that column, and again, after the second. There is no remainder, as we take away in succession the three component parts of the cube of $a + b + c$.

2. Required the cube root of $a^6 - 6a^5 + 21a^4 - 44a^3 + 63a^2 - 54a + 27$.

Here we arrange the given quantity and two ciphers on the same line; and taking a^2 for the first term of the root we place it in the first column, and then form the first and second columns in the manner already shewn; the first trial divisor is $3a^4$, which, with $-6a^5$ gives $-2a$ for the new term of the root; adding this new term in the first column, multiplying the sum by it, and adding in the second column, the complete divisor is found to be $3a^4 - 6a^3 + 4a^2$. The new trial divisor is $3a^4 - 12a^3 + 12a^2$ which gives 3 for the next term of the root, and so on, as already described. The root is $a^2 - 2a + 3$.

$$\begin{array}{l}
0\\
a^2\\
\underline{a^2}\\
a^2\\
\overline{2a^2}\\
a^2\\
\overline{3a^2}\\
\qquad -2a\\
\overline{3a^2-2a}\\
\qquad -2a\\
\overline{3a^2-4a}\\
\qquad -2a\\
\overline{3a^2-6a}\\
\qquad\qquad +3\\
\overline{3a^2-6a+3}
\end{array}
\qquad
\begin{array}{l}
0\\
a^4\\
2a^4\\
\overline{3a^4}\\
\qquad -6a^3+4a^2\\
\overline{3a^4-6a^3+4a^2}\\
\qquad -6a^3+8a^2\\
\overline{3a^4-12a^3+12a^2}\\
\qquad\qquad\qquad 9a^2-18a+9\\
\overline{3a^4-12a^3+21a^2-18a+9}
\end{array}
\qquad
\begin{array}{r}
a^2-2a+3\\
\overline{a^6-6a^5+21a^4-44a^3+63a^2-54a+27}\\
a^6\qquad\qquad\qquad\qquad\qquad\qquad\\
\overline{-6a^5+21a^4-44a^3}\qquad\qquad\quad\\
-6a^5+12a^4-8a^3\qquad\qquad\quad\\
\overline{9a^4-36a^3+63a^2-54a+27}\\
9a^4-36a^3+63a^2-54a+27\\
\overline{0\qquad\qquad\qquad}
\end{array}$$

This and similar questions may be very conveniently worked by the method of detached coefficients, as follows :—

$$\begin{array}{lll}
 & & \overline{1-2+3} \\
0 & 0 & 1-6+21-44+63-54+27 \\
\underline{1} & \underline{1} & \underline{1} \\
1 & 1 & -6+21-44 \\
\underline{1} & \underline{2} & \underline{-6+12-8} \\
2 & 3 & 9-36+63-54+27 \\
\underline{1} & \underline{-6+4} & \underline{9-36+63-54+27} \\
3 & 3-6+4 & 0 \\
\underline{-2} & \underline{-6+8} & \\
3-2 & 3-12+12 & \\
\underline{-2} & \underline{9-18+9} & \\
3-4 & 3-12+21-18+9 & \\
\underline{-2} & & \\
3-6 & & \\
\underline{+3} & & \\
3-6+3 & &
\end{array}$$

Attaching the powers of a the root is a^2-2a+3.

The square roots and the cube roots of numbers are extracted in the same manner, and the theory of the process is the same. Numbers of two places of figures may be expressed as binomials, those of three places as trinomials, &c. Thus $36=30+6$, and $782=700+80+2$; and their squares and cubes may therefore be expressed in the same forms as those of binomial and trinomial quantities. Hence the powers being given, the converse process for finding the root will be the same as that above described.

99. Since (Art. 86) any root may be expressed as a power with a fractional index, we may extend the application of the binomial theorem to the extraction of roots, by taking for the index n, the fraction which expresses the root to be extracted: and it might be shewn that in this case, as in Example 2, Art. 92, the series will never terminate.

1. Expand $\sqrt{(a^2+x^2)}$ by the binomial theorem.

Here, since $\sqrt{(a^2+x^2)} = (a^2+x^2)^{\frac{1}{2}}$, we must put for *a*, *b*, and *n*, in the general formula, a^2, x^2, and $\frac{1}{2}$: then

$$(a^2+x^2)^{\frac{1}{2}} = (a^2)^{\frac{1}{2}} + \tfrac{1}{2}(a^2)^{-\frac{1}{2}}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})}{2}(a^2)^{-\frac{3}{2}}x^4 +$$
$$\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{2\,.\,3}(a^2)^{-\frac{5}{2}}x^6 \text{ \&c.}$$

or, $$\sqrt{(a^2+x^2)} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} \text{ \&c.}$$

This is the same series which would be found by actual evolution. See Exercise 7.

2. Find the value of $\sqrt[3]{9}$ in a series.

Here we have $\sqrt[3]{9} = (8+1)^{\frac{1}{3}}$; then

$$(8+1)^{\frac{1}{3}} = 8^{\frac{1}{3}} + \tfrac{1}{3}\,.\,8^{-\frac{2}{3}} + \frac{\frac{1}{3}(-\frac{2}{3})}{2}8^{-\frac{5}{3}} - \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{2\,.\,3}8^{-\frac{8}{3}}$$
$$+ \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{2\,.\,3\,.\,4}8^{-\frac{11}{3}} \text{ \&c.}$$

or,

$$\sqrt[3]{9} = 2 + \tfrac{1}{3}\frac{1}{(8^{\frac{1}{3}})^2} - \frac{2}{3\,.\,3\,.\,2}\cdot\frac{1}{(8^{\frac{1}{3}})^5} + \frac{2\,.\,5}{3\,.\,3\,.\,3\,.\,6}\cdot\frac{1}{(8^{\frac{1}{3}})^8}$$
$$- \frac{2\,.\,5\,.\,8}{3\,.\,3\,.\,3\,.\,3\,.\,6\,.\,4}\cdot\frac{1}{(8^{\frac{1}{3}})^n} \text{ \&c.}$$

whence

$$\sqrt[3]{9} = 2 + \frac{1}{3\,.\,2^2} - \frac{1}{3\,.\,6\,.\,2^4} + \frac{5}{3\,.\,6\,.\,9\,.\,2^7} - \frac{5\,.\,8}{3\,.\,6\,.\,9\,.\,12\,.\,2^{16}} \text{ \&c.}$$

EXERCISES.

Find the square roots of the following quantities :—

1. $25a^2 + 60a + 36$.
2. $9x^2 + 24xy + 16y^2$.

3. $a^2 - ax + \frac{1}{4}a^2$.
4. $4x^4 - 16x^3 + 24x^2 - 16x + 4$.
5. $4a^4 + 12a^3x + 13a^2x^2 + 6ax + x^4$.
6. $2 + x + \dfrac{1}{x}$.
7. $a^2 + x^2$.
8. $25x^4 - 30ax^3 + 49a^2x^2 - 24a^3x + 16a^4$.
9. $x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{16}$.
10. $a^{2m}x^{2n} + 10ca^{2m-2}x^{2n+1} - 6a^{m+1}x^{n-1} + 25c^2a^{2m-4}x^{2n+2} -$
    $30ca^{m-1}x^n + \dfrac{9a^2}{x^2}$.
11. $x^6 - 6ax^5 + 15a^2x^4 - 20a^3x^3 + 15a^4x^2 - 6a^5x + a^6$.
12. $p^2 + 2pqx + (2pr + q^2)x^2 + 2(ps + qr)x^3 + (2qs + r^2)x^4 + 2rsx^5$
    $+ s^2x^6$.
13. Find the condition on which the quantity $a^2x^2 + px + q^2$ shall be a complete square.

Find the cube roots of the following quantities:

14. $8x^3 - 60x^2 + 150x + 125$.
15. $\dfrac{a^3c^3}{b^3}x^6 - \dfrac{3a^2c}{b}x^5 + \dfrac{3ab}{c}x^4 - \dfrac{b^3}{c^3}x^3$.
16. $(a+1)^{6n}x^3 + 6ca^p(a+1)^{4n}x^2 + 12c^2a^{2p}(a+1)^{2n}x + 8c^3a^{3p}$.
17. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.
18. $27x^6 - 54x^5 + 63x^4 - 44x^3 + 21x^2 - 6x + 1$.
19. $8x^6 + 48ax^5 + 60a^2x^4 - 80a^3x^3 - 90a^4x^2 + 108a^5x - 27a^6$.
20. $1 + 6x + 21x^2 + 56x^3 + 111x^4 + 174x^5 + 219x^6 + 204$
    $x^7 + 144x^8 + 64x^9$.
21. $a^{3m} + 12a^{m+2}x^{2n} - 8a^3x^{3n} - 6a^{2m+1}x^n$.
22. $27x^6 + 54x^5 + 225x^4 + 260x^3 + 525x^2 + 294x + 343$.
23. $x^6 - 3x^5 + 9x^4 - 13x^3 + 18x^2 - 12x + 8$.
24. Find what c and d must be in order that the quantity $a^3x^3 + bx^2 + cx + d$ may be a complete cube.
25. Find the relations which exist among the quantities m, n, p, q, when $mx^3 + nx^2 + px + q$ is a complete cube.

Expand the following quantities into series by the binomial theorem :—

26. $\sqrt{(b^2+x)}$.

27. $\dfrac{1}{\sqrt{(b^2+x)}}$.

28. $\sqrt{2}$.

29. $\sqrt{(1+x)}$.

30. $\sqrt[n]{(1+x)}$.

31. $\sqrt[3]{(a+x)^2}$.

32. $\sqrt[3]{7}$ or $\sqrt[3]{(8-1)}$.

33. $\dfrac{a}{\sqrt{(c^2+x^2)}}$.

34. $(a^2-x^2)^{-\frac{1}{2}}$.

35. $(1+x)^{-\frac{1}{n}}$.

36. $\sqrt[5]{2}$.

37. $\sqrt{(a-x)}$.

38. $\sqrt[3]{(a+x)^4}$.

X.—SURDS.

Art. 100. Any root of a quantity which cannot be exactly evolved, is called a *surd*, or *radical*, or an *irrational quantity:* all other quantities are called *rational.* Thus, $\sqrt{8}$, $\sqrt[3]{9}$, are irrational quantities; but $\sqrt[3]{8}$, $\sqrt{9}$, are rational: also, $\sqrt[3]{(2a)}$, $\sqrt{(a^2+x^2)}$, $\sqrt{6}$, $\sqrt[3]{(10b^2)}$, &c. are surds. Surds are distinguished from one another by the degree of the root; thus $\sqrt{a}$, $\sqrt{8}$, are *quadratic* surds, $\sqrt[3]{9}$, $\sqrt[3]{\overline{2a}}$ are *cubic* surds, &c.

101. *The square root of a rational quantity cannot be partly rational, and partly irrational.*

For, if possible, let $\sqrt{a}=b+\sqrt{c}$; then (Art. 31), $a=b^2+2b\sqrt{c}+c$, and (Art. 28) $a-b^2-c=2b\sqrt{c}$: hence (Art. 30), $\dfrac{a-b^2-c}{2b}=\sqrt{c}$, a rational quantity equal to an irrational, which is impossible.

102. *In any expression of the form $a+\sqrt{b}=c+\sqrt{d}$, the rational parts are equal to one another, and also the irrational parts.*

For, if a be not equal to c, let $a=c+x$, then $c+x+\sqrt{b}=c+\sqrt{d}$; therefore (Art. 28) $x+\sqrt{b}=\sqrt{d}$, which is impossible (Art. 101).

103. If $\sqrt{(a+\sqrt{b})}=c+\sqrt{d}$, then will $\sqrt{(a-\sqrt{b})}=c-\sqrt{d}$.

Since (Art. 31) $a+\sqrt{b}=c^2+2c\sqrt{d}+d$, we have (Art. 102) $a=c^2+d$, and $\sqrt{b}=2c\sqrt{d}$; whence (Art. 28) $a-\sqrt{b}=c^2-2c\sqrt{d}+d$, and (Art. 31, 97), $\sqrt{(a-\sqrt{b})}=c-\sqrt{d}$.

Also if $\sqrt[3]{(a+\sqrt{b})}=x+\sqrt{y}$, then $\sqrt[3]{(a-\sqrt{b})}=x-\sqrt{y}$;

Suppose $\sqrt[3]{(a+\sqrt{b})}=x+\sqrt{y}$, then

$$a+\sqrt{b}=x^3+3x^2\sqrt{y}+3xy+y\sqrt{y};$$

therefore (Art. 102),

$$a=x^3+3xy, \text{ and } \sqrt{b}=3x^2\sqrt{y}+y\sqrt{y};$$

hence

$$a-\sqrt{b}=x^3-3x^2\sqrt{y}+3xy-y\sqrt{y},$$

and therefore

$$\sqrt[3]{(a-\sqrt{b})}=x-\sqrt{y}.$$

104. *If the product of two quadratic surds is a rational quantity, one of the surds must be a rational multiple of the other.*

Let $\sqrt{x}\times\sqrt{y}=m$, a rational quantity; then $\sqrt{y}=\frac{m}{\sqrt{x}}=\frac{m}{x}\sqrt{x}$; and since m and x are rational, $\sqrt{y}$ is a rational multiple of $\sqrt{x}$. Hence, if one quadratic surd is not a rational multiple of another, their product cannot be a rational quantity.

105. *One quadratic surd cannot be made up of two others, not having the same irrational part.*

For, if possible, let $\sqrt{x}=\sqrt{m}+\sqrt{n}$; then $x=m+n+2\sqrt{mn}$, and $\frac{x-m-n}{2}=\sqrt{mn}$, a rational quantity equal to an irrational, which is absurd.

106. *The difference of any two equal integral powers is always divisible by the difference of their roots; that is, $x^n - y^n$ is divisible by $x - y$ when n is a positive integer.*

Let $x - y = d$, therefore (Art. 27) $x = d + y$; then $x^n - y^n = (d + y)^n - y^n$. Expanding $(d + y)^n$ by the binomial theorem, we have

$$(d + y)^n = d^n + nd^{n-1}y + n\frac{n-1}{2}d^{n-2}y^2 + \&c. \ldots + ndy^{n-1} + y^n;$$

therefore (Art. 28),

$$(d + y)^n - y^n = d^n + nd^{n-1}y + n\frac{n-1}{2}d^{n-2}y^2 + \&c. \ldots\ldots\ldots + ndy^{n-1}.$$

Now, d will obviously divide all the terms of this latter part; therefore, $(d + y)^n - y^n$, or $x^n - y^n$, is also divisible by d, that is, by $x - y$. *See Note*, p. 48.

107. *The difference of two equal powers is divisible by the sum of the roots, when the exponent is even.*

Let $x + y = s$, therefore (Art. 28) $x = s - y$; then $x^n - y^n = (s - y)^n - y^n$; and if $(s - y)^n$ be expanded, its last term, y^n, will be positive (Art. 92, and hyp.): then subtracting y^n as before, the remaining terms will be divisible by s; therefore $(s - y)^n - y^n$, or $x^n - y^n$, is also divisible by s, that is, by $x + y$.

108. *The sum of two equal powers is divisible by the sum of the roots, when the index is odd.*

Let $x + y = s$, therefore $x = s - y$, and $x^n + y^n = (s - y)^n + y^n$. In this the last term of the expansion will be negative: hence, if y^n be added (Art. 27), the terms of the expansion will be divisible by s; and $(s - y)^n + y^n$, or $x^n + y^n$, will also be divisible by s, that is, by $x + y$.*

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* These propositions may also be established as follows:—
The quantity $x^n - a^n$ may be put under the form

109. A rational quantity may be reduced to the form of a given surd by *raising the quantity to the power expressed by the index of the surd, and then indicating the extraction of the same root.* In the same way, the rational coefficient of a surd may be taken under the radical sign.

Thus, $4a^2 = \sqrt[3]{(64a^6)}$; $a + x = (a + x)^{\frac{n}{n}}$; $a\sqrt{x} = \sqrt{a^2} \times \sqrt{x} = \sqrt{(a^2x)}$ Art. 95.

Introduce the coefficient in the expression $4\sqrt{5}$, under the radical sign.

Here, $4\sqrt{5} = \sqrt{16} \times \sqrt{5} = \sqrt{80}$ (Art. 95).

Express $a^{\frac{1}{2}}$ in such a form that it shall have the same index as $x^{\frac{2}{3}}$.

Here the index must be such that when multiplied by $\frac{2}{3}$, it shall produce $\frac{1}{2}$; therefore $\frac{1}{2} \div \frac{2}{3} = \frac{3}{4}$, is the required index, and $(a^{\frac{3}{4}})^{\frac{2}{3}}$, the form required.

$$x^n - ax^{n-1} + ax^{n-1} - a^2x^{n-2} + a^2x^{n-2} - \dots$$
$$\dots - a^{n-1}x + a^{n-1}x - a^n,$$

in which the two first terms are the same as $x^{n-1}(x - a)$, the next two as $ax^{n-2}(x - a)$; the two last the same as $a^{n-1}(x - a)$, and the two before these as $a^{n-2}x(x - a)$. Since $x - a$ is thus shewn to be a factor of every term; the above expression, or its equal $x^n - a^n$, must be divisible by $x - a$. The quotient is

$$x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-3}x^2 + a^{n-2}x + a^{n-1}.$$

If we now change the sign of a, putting $x + a$ for $x - a$, $x^n - a^n$ will continue the same when n is *even;* because (Arts. 90, 92) the nth power of a, and the nth power of $-a$, are then the same; but the sign will change when n is odd; for an odd power of $-a$ being negative, $-a^n$ will then become a^n and $x^n - a^n$, $x^n + a^n$. Hence, when n is odd, $x^n + a^n$ is divisible by $x + a$, but the signs of the terms will differ; and when n is even, $x^n - a^n$ is divisible by $x + a$ as well as by $x - a$. Calling then the indices $2n$ and $2n + 1$, we have

$$\frac{x^{2n+1} + a^{2n+1}}{x + a} = x^{2n} - ax^{2n-1} + a^2x^{2n-2} - \dots + a^{2n},$$

$$\frac{x^{2n} - a^{2n}}{x + a} = x^{2n-1} - ax^{2n-2} + a^2x^{2n-3} - \dots - a^{2n-1},$$

$$\frac{x^{2n} - a^{2n}}{x - a} = x^{2n-1} + ax^{2n-2} + a^2x^{2n-3} + \dots + a^{2n-1}.$$

But $x^{2n} + a^{2n}$ is not divisible by either $x + a$ or $x - a$.

110. To reduce a surd to its simplest form; *Resolve it, if possible, into two factors, one of which shall be the greatest power contained in it, corresponding to the surd-index, and take the root of each factor.* If the given surd be in a fractional form, *multiply both terms of the fraction by such a quantity as will make the denominator a complete power, corresponding to the root, and take the root of each term.*

Thus, $\sqrt[3]{135} = \sqrt[3]{(27 \times 5)} = \sqrt[3]{27} \times \sqrt[3]{5} = 3\sqrt[3]{5}$.

In reducing $2\sqrt[3]{\frac{2}{3}}$ to its simplest form, we multiply both terms by $\sqrt[3]{9}$, thus making the denominator a complete cube; we thus get $2\sqrt[3]{\frac{2}{3}} = 2\sqrt[3]{\frac{18}{27}} = 2\sqrt[3]{(\frac{1}{27} \times 18)} = \frac{2}{3}\sqrt[3]{18}$.

To simplify $(4a^5b^2 - 20a^3b^3 + 25ab^4)^{\frac{1}{2}}$, we have for factors $(2a^2 - 5b)^2$ and ab^2; the reduced form therefore is,

$$[(2a^2 - 5b)^2 \times ab^2]^{\frac{1}{2}} = (2a^2 - 5b)ba^{\frac{1}{2}}.$$

111. To reduce surds having different indices to equivalent ones having a common index; *Reduce the indices to a common denominator, involve the quantities to the power denoted by the numerator, and indicate the extraction of the root expressed by the denominator.*

The reason of the rule is manifest from Arts. 71, 86.

Thus $2^{\frac{1}{2}}$ and $4^{\frac{1}{3}}$ will be reduced to surds expressing the same root by reducing $\frac{1}{2}$ and $\frac{1}{3}$ to the same denominator; we thus get $2^{\frac{3}{6}}$ and $4^{\frac{2}{6}}$; but $2^{\frac{3}{6}} = (2^3)^{\frac{1}{6}} = 8^{\frac{1}{6}}$ and $4^{\frac{2}{6}} = (4^2)^{\frac{1}{6}} = 16^{\frac{1}{6}}$ so that $8^{\frac{1}{6}}$ and $16^{\frac{1}{6}}$ are the surds required.

112. To add or subtract surds; *Reduce them, if necessary, to their simplest forms, and proceed as in adding or subtracting rational quantities.*

The reason of the rule is manifest from Arts. 23, 32, 110.

Thus, $a\sqrt{x} \pm b\sqrt{x} = (a \pm b)\sqrt{x}$; $\sqrt{45} \pm \sqrt{20} = 3\sqrt{5} \pm 2\sqrt{5} = 5\sqrt{5}$, or $\sqrt{5}$; $(3a^2b)^{\frac{1}{2}} + (48x^2b)^{\frac{1}{2}} = (a + 4x)\sqrt{3b}$, and $2\sqrt{48} \pm 9\sqrt[3]{108} = 8\sqrt{3} + 27\sqrt[3]{4}$, the terms not incorporating, as the indices of the surds are different.

113. To multiply or divide surds; *Reduce them, if necessary, to equivalent ones having a commen index, and proceed as in multiplying rational quantities.*

Thus, $5\sqrt[3]{a} \times 3\sqrt[4]{b} = 5a^{\frac{1}{3}} \times 3b^{\frac{1}{4}} = 5a^{\frac{4}{12}} \times 3b^{\frac{3}{12}} = 5(a^4)^{\frac{1}{12}} \times 3(b^3)^{\frac{1}{12}}$ (Art. 111); hence, Art. 95, the product is $15(a^4b^3)^{\frac{1}{12}}$, or $15\sqrt[12]{(a^4b^3)}$; so also $(a+x)^{\frac{1}{2}} \times (a-x)^{\frac{1}{3}} = (a+x)^{\frac{3}{6}} \times (a-x)^{\frac{2}{6}} = \{(a+x)^3(a-x)^2\}^{\frac{1}{6}}$, or $\sqrt[6]{\{(a+x)^3(a-x)^2\}}$. But if the surds have a common index, it is only necessary to place it over the product of the surds. Thus $(a+x)^{\frac{1}{2}} \times (a-x)^{\frac{1}{2}} = (a^2-x^2)^{\frac{1}{2}}$.

Again, since (Art. 95) $\sqrt[n]{(ab)} = \sqrt[n]{a} \times \sqrt[n]{b}$, it follows from the nature of division, that $\sqrt[n]{(ab)} \div \sqrt[n]{a} = \sqrt[n]{b}$; so also $\sqrt[n]{a^m} \div \sqrt[n]{b^p} = \sqrt[n]{\frac{a^m}{b^p}}$, and $(a^2-x^2)^{\frac{1}{2}} \div (a-x)^{\frac{1}{2}} = (a+x)^{\frac{1}{2}}$. But if the surds have not a common index, they must be reduced by Art. 111. Thus $\sqrt[m]{a} \div \sqrt[n]{b} = a^{\frac{1}{m}} \div b^{\frac{1}{n}} = a^{\frac{n}{mn}} \div b^{\frac{m}{mn}} = (a^n)^{\frac{1}{mn}} \div (b^m)^{\frac{1}{mn}} = \left(\frac{a^n}{b^m}\right)^{\frac{1}{mn}}$ or $\sqrt[mn]{\frac{a^n}{b^m}}$. So also, $(a+x)^{\frac{1}{2}} \div (a-x)^{\frac{1}{3}} = (a+x)^{\frac{3}{6}} \div (a-x)^{\frac{2}{6}} = \left\{\frac{(a+x)^3}{(a-x)^2}\right\}^{\frac{1}{6}}$ or $\sqrt[6]{\frac{(a+x)^3}{(a-x)^2}}$.

114. To extract the square root of a binomial surd.

Let the binomial surd be $a \pm \sqrt{b}$, and let

$$\sqrt{(a \pm \sqrt{b})} = \sqrt{x} \pm \sqrt{y}:$$

hence (Arts. 31, 87), we have

$$a \pm \sqrt{b} = x \pm 2\sqrt{xy} + y.$$

Now (Art. 102),

$$a = x + y,\ \sqrt{b} = 2\sqrt{xy};$$

therefore (Arts. 31, 87),

$$a^2 = x^2 + 2xy + y^2, \text{ and } b = 4xy:$$

hence (Art. 28),

$$a^2 - b = x^2 - 2xy + y^2$$

and (Arts. 31, 97),

$$\sqrt{(a^2 - b)} = x - y;$$

But

$$a = x + y;$$

therefore (Arts. 27, 28),

$$a + \sqrt{(a^2 - b)} = 2x, \quad a - \sqrt{(a^2 - b)} = 2y;$$

and (Art. 30),

$$\frac{a + \sqrt{(a^2 - b)}}{2} = x, \quad \frac{a - \sqrt{(a^2 - b)}}{2} = y.$$

Hence (Art. 31)

$$\sqrt{x} \pm \sqrt{y} \text{ or } \sqrt{(a \pm \sqrt{b})} = \sqrt{\frac{a + \sqrt{(a^2 - b)}}{2}} \pm \sqrt{\frac{a - \sqrt{(a^2 - b)}}{2}}.$$

Required the square root of $7 - 2\sqrt{10}$.

Assuming $\sqrt{x} - \sqrt{y} = \sqrt{(7 - 2\sqrt{10})}$, and following the steps of the preceding process, we have

$$x + y - 2\sqrt{xy} = 7 - 2\sqrt{10}$$

hence $x + y = 7$, and $2\sqrt{xy} = 2\sqrt{10}$

whence $x^2 + 2xy + y^2 = 49$

and $4xy = 40;$

therefore $x^2 - 2xy + y^2 = 9,$

and $x - y = 3;$

but $x + y = 7;$

therefore $2x = 10$, or $x = 5;$

and $2y = 4$, or $y = 2,$

hence the required root is

$$\sqrt{x} - \sqrt{y} = \sqrt{5} - \sqrt{2}.$$

The square root of $4 + 2\sqrt{3}$ will be found by substituting for a and $\sqrt{b}$ in the general formula, 4 and $2\sqrt{3}$; we thus have

$$\sqrt{(4 + 2\sqrt{3})} = \sqrt{\frac{4 + \sqrt{(16 - 12)}}{2}} + \sqrt{\frac{4 - \sqrt{(16 - 12)}}{2}}$$

$$= \sqrt{3} + 1.$$

Required the square root of $\sqrt{27} + \sqrt{24}$.
This may be worked by substitution, as in Ex. 2; if after the manner of Ex. 1, it may be first simplified, thus:

$$\sqrt{27} + \sqrt{24} = \sqrt{9 \times 3} + \sqrt{8 \times 3} = \sqrt{3}(3 + \sqrt{8});$$

therefore $\sqrt{(\sqrt{27} + \sqrt{24})} = \sqrt[4]{3}\ \sqrt{(3 + \sqrt{8})}$.
Now, $\sqrt{(3 + \sqrt{8})}$ found as in Ex. 1, is $1 + \sqrt{2}$; the root therefore is $\sqrt[4]{3}(1 + \sqrt{2}) = \sqrt[4]{3} + \sqrt[4]{12}$, since $\sqrt{2} = \sqrt[4]{4}$.

115. We are sometimes able to extract the square root of a quantity of the form

$$a + \sqrt{b} + \sqrt{c} + \sqrt{d}.$$

Assume $\sqrt{(a + \sqrt{b} + \sqrt{c} + \sqrt{d})} = \sqrt{x} + \sqrt{y} + \sqrt{z}$.
Squaring we have

$$a + \sqrt{b} + \sqrt{c} + \sqrt{d} = x + y + z + 2\sqrt{(xy)} + 2\sqrt{(yz)} + 2\sqrt{(xz)}.$$

If, then, we make $2\sqrt{(xy)} = \sqrt{b}$, $2\sqrt{(yz)} = \sqrt{c}$, and $2\sqrt{(xz)} = \sqrt{d}$, and find that the values thus given make $a = x + y + z$, we shall have the root required.

Thus, if the square root of $6 + 2\sqrt{2} + 2\sqrt{3} + 2\sqrt{6}$, be required, put

$$\sqrt{(6 + 2\sqrt{2} + 2\sqrt{3} + 2\sqrt{6}} = \sqrt{x} + \sqrt{y} + \sqrt{z},$$

take the square of both expressions, and make,

$$2\sqrt{(xy)} = 2\sqrt{2}, \quad 2\sqrt{(yz)} = 2\sqrt{3}, \quad 2\sqrt{(xz)} = 2\sqrt{6},$$

then,

$$\sqrt{(xy)} = \sqrt{2}, \qquad \sqrt{(yz)} = \sqrt{3}, \qquad \sqrt{(xz)} = \sqrt{6}.$$

And if we multiply together the first two of these, and divide by the third, we find $y = 1$; but $xy = 2$; therefore, $x = 2$; also, $z = 3$; and these three values satisfy the condition, $x + y + z = 6$. Hence, the required root is $\sqrt{2} + \sqrt{1} + \sqrt{3}$, or $1 + \sqrt{2} + \sqrt{3}$ (Art. 103).

116. In certain cases, the cube root of a binomial $a \pm \sqrt{b}$ may be found.

Suppose $\sqrt[3]{(a+\sqrt{b})}=x+\sqrt{y}$.
Then (Art. 103) $\sqrt[3]{(a-\sqrt{b})}=x-\sqrt{y}$.
Taking the product, we have $\sqrt[3]{(a^2-b)}=x^2-y$.

Assume now that a^2-b is a perfect cube, and denote it by c^3: then $c=x^2-y$; but (Art. 102) $a=x^3+3xy$; and $y=x^2-c$; whence $a=x^3+3x(x^2-c)$ and $a=4x^3-3cx$, an expression from which x can only be found by trial. The method is applicable only when a^2-b is a perfect cube, as it is upon this assumption that the investigation proceeds.

Thus, let the cube root of $7+5\sqrt{2}$ be required.

Assume

$\sqrt[3]{(7+5\sqrt{2})}=x+\sqrt{y}$; then $\sqrt[3]{(7-5\sqrt{2})}=x-\sqrt{y}$; and taking the product, $\sqrt[3]{(49-50)}=x^2-y$; whence $-1=x^2-y$, and $y=x^2+1$. Now (Art. 103) $7=x^3+3xy$ $=x^3+3x(x^2+1)=4x^3+3x$; also $4x^3+3x=7$, when $x=1$; and if $x=1$, the expression $x^3+3xy=7$, gives $1+3y=7$; so that $y=2$; and hence the required cube root is $1+\sqrt{2}$. This result may be verified by raising $1+\sqrt{2}$ to the third power.

117. To find such multipliers as will make binomial surds rational. Performing the actual division which has been proved possible, in Arts. 106, 107, 108, we have

$$\frac{x^n-y^n}{x-y}=x^{n-1}+x^{n-2}y+x^{n-3}y^2+x^{n-4}y^3\ \&c.$$
$$\frac{x^n-y^n}{x+y}=x^{n-1}-x^{n-2}y+x^{n-3}y^2-x^{n-4}y^3\ \&c.$$
$$\frac{x^n+y^n}{x+y}=x^{n-1}-x^{n-2}y+x^{n-3}y^2-x^{n-4}y^3\ \&c.$$

And it is plain that the first series will terminate at the nth term, whether n be even or odd; the second only when n is even, and the third only when n is odd.

Now, let $x^n = a$, $y^n = b$; therefore $x = \sqrt[n]{a}$, $y = \sqrt[n]{b}$, $x^{n-1} = \sqrt[n]{a^{n-1}}$, $x^{n-2} = \sqrt[n]{a^{n-2}}$, &c.; and $y^2 = \sqrt[n]{b^2}$, $y^3 = \sqrt[n]{b^3}$, &c.

Substituting these values in the above series, we have

$$\frac{a-b}{\sqrt[n]{a}-\sqrt[n]{b}} = \sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b^2} + \sqrt[n]{a^{n-4}b^3} + \text{\&c.}$$

$$\frac{a-b}{\sqrt[n]{a}+\sqrt[n]{b}} = \sqrt[n]{a^{n-1}} - \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b^2} - \sqrt[n]{a^{n-4}b^3} \text{ \&c.}$$

$$\frac{a+b}{\sqrt[n]{a}+\sqrt[n]{b}} = \sqrt[n]{a^{n-1}} - \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b^2} - \sqrt[n]{a^{n-4}b^3} \text{ \&c.}$$

Hence, by the nature of division, if each of these irrational divisors be multiplied by the corresponding irrational quotient, the corresponding rational dividend will be reproduced; and, therefore, the first of these quotients is the multiplier, which will render rational a surd of the form $\sqrt[n]{a}-\sqrt[n]{b}$; and the second and third quotients are the multipliers, which will make a surd of the form $\sqrt[n]{a}+\sqrt[n]{b}$ rational; the binomial which is produced having its second term *negative* or *positive*, according as n is even or odd.*

Required the multiplier which will make $\sqrt[3]{9}+\sqrt[3]{10}$ rational.

Here $a = 9$, $b = 10$, $n = 3$; then, since n is odd, and the second term positive, the third form applies to this case;

---

* In many cases, multipliers may be more readily found than by the above general method. Thus, it is plain that a surd of the form $\sqrt[n]{a}$ will be made rational by the multiplier $b\sqrt{a^{n-1}}$; for $\sqrt[n]{a}\times\sqrt[n]{a^{n-1}} = \sqrt[n]{a^n} = a$. Also, since the product of the sum and difference of two quantities is the difference of their squares (Note p. 32), it follows that $\sqrt{a}\pm\sqrt{b}$ will be made rational by the multiplier $\sqrt{a}\mp\sqrt{b}$. Again, since $(a\mp b)\times(a^2\pm ab+b^2) = a^3\mp b^3$, a surd of the form $\sqrt[3]{a}\mp\sqrt[3]{b}$ will be made rational by the multiplier $\sqrt[3]{a^2}\pm\sqrt[3]{ab}+\sqrt[3]{b^2}$. Several of the exercises in the text may be worked by this method.

therefore we have $\sqrt[n]{a^{n-1}} - \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b^2}$ &c. $= \sqrt[3]{81} - \sqrt[3]{90} + \sqrt[3]{100}$. The actual multiplication is as under :

$$\begin{array}{ll}
\text{multiplier,} & \sqrt[3]{81} - \sqrt[3]{90} + \sqrt[3]{100} \\
\text{given surd,} & \sqrt[3]{9} + \sqrt[3]{10} \\
\hline
 & 9 - \sqrt[3]{810} + \sqrt[3]{900} \\
 & \qquad \sqrt[3]{810} - \sqrt[3]{900} + 10 \\
\hline
\text{rational product,} & 9 + 10 = 19
\end{array}$$

The multiplier might have been more readily found, by applying the principle in the Note.

Reduce $\dfrac{3}{\sqrt{5} - \sqrt{2}}$ to a fraction whose denominator shall be rational.

Here the multiplier is plainly $\sqrt{5} + \sqrt{2}$; then multiplying both terms by this, we have

$$\frac{3\sqrt{5} + 3\sqrt{2}}{5 - 2} = \sqrt{5} + \sqrt{2}.$$

What multiplier will make $3^{\frac{1}{4}} - 4^{\frac{1}{4}}$ rational ?

The multiplier, found by the general formula, is $27^{\frac{1}{4}} + 36^{\frac{1}{4}} + 48^{\frac{1}{4}} + 64^{\frac{1}{4}}$, but by the principle in the Note, we may use the two multipliers, $3^{\frac{1}{4}} + 4^{\frac{1}{4}}$ and $3^{\frac{1}{2}} + 4^{\frac{1}{2}}$, in succession. In fact, the above multiplier is the product of these two.

EXERCISES.

Reduce

1. $3x^2$ to the form of the square root.
2. $2a^2b$ „ cube „
3. $-3a$ „ „ „
4. $a^2b^{\frac{1}{2}}$ „ nth „
5. $-2n^2$ „ 5th „
6. $3\sqrt[3]{x}$ „ cube „
7. $\sqrt{a}$ „ 3d „

8. $\frac{1}{4}\sqrt{8x}$ to the form of the square root.
9. $2\sqrt[3]{3}$,, ,, . ,,
10. $(a-x)^{\frac{2}{3}}$,, exponent $\frac{5}{6}$.
11. $a^{\frac{2}{5}}$,, ,, $\frac{7}{10}$.

Introduce

12. The coefficient a under the exponent in the expression $ax^{\frac{p}{n}}$.
13. ,, 27 ,, $27(a+b)^{\frac{2}{3}}$.
14. $\sqrt{a}$,, ,, $\sqrt[3]{(a+x)}$.

Resolve; simplify;

15. $(9a^4x)^{\frac{1}{2}}$
16. $4\sqrt[3]{(8a^5+16a^3)}$
17. $\sqrt{75}$
18. $\sqrt{405}$
19. $\sqrt{147}$
20. $\sqrt[3]{162}$
21. $\sqrt{\frac{3}{4}}$
22. $\frac{3}{4}\sqrt{\frac{1}{2}}$
23. $\frac{1}{3}\sqrt{\frac{5}{9}}$
24. $\sqrt{\frac{4}{3}}$
25. $\sqrt{\frac{a}{b}}$
26. $\left(\frac{a^3x-4a^2x^2+4ax^3}{n^2p^2}\right)^{\frac{1}{2}}$
27. $(72x+108y)^{\frac{1}{2}}$
28. $\left(\frac{a^3b^2}{cd^2}-\frac{a^2b^3}{c^2d}\right)^{\frac{1}{2}}$
29. $\frac{2x}{3a}(108x^{3n})^{\frac{1}{3}}$
30. $(3a^2c+6abc+3b^2c)^{\frac{1}{2}}$
31. $(2ny^2-4ny+2n)^{\frac{1}{2}}$
32. $(a^2b)^{\frac{1}{2}}$
33. $(8a)^{\frac{1}{3}}$
34. $(a^3+a^3b^2)^{\frac{1}{3}}$
35. $(2x^3y^2+bx^5y)^{\frac{1}{3}}$
36. $\sqrt{\frac{1}{3}}$
37. $\sqrt{\frac{2}{7}}$
38. $\frac{1}{3}\sqrt[3]{\frac{4}{9}}$
39. $\left(\frac{a^2x-2ax^2+x^3}{a^2+2ax+x^2}\right)^{\frac{1}{2}}$
40. $\frac{1}{3}\sqrt[3]{\frac{16}{81}}$

Reduce to a common index,

41. $a^{\frac{1}{2}}$ and $a^{\frac{2}{3}}$
42. $2^{\frac{1}{2}}$ and $3^{\frac{1}{5}}$
43. $2a^{\frac{2}{3}}$ and $3a^{\frac{3}{4}}$
44. a^2 and $b^{\frac{2}{3}}$
45. $5^{\frac{2}{3}}$ and $6^{\frac{4}{5}}$
46. $2\sqrt[3]{3}$ and $3\sqrt{2}$.
47. $(a+x)^{\frac{1}{2}}$ and $(b-x)^{\frac{1}{4}}$
48. x^n and $y^{\frac{1}{n}}$
49. $\sqrt{\frac{1}{2}}$ and $\sqrt[5]{\frac{3}{4}}$
50. $ax^{\frac{1}{n}}$ and $by^{\frac{1}{m}}$

Find the sum or difference, according to the sign,

51. $\sqrt{45} + \sqrt{20}$
52. $2\sqrt[3]{5} + 5\sqrt{6}$
53. $\sqrt{\frac{25}{8}} + \sqrt{\frac{4}{27}} + \sqrt{\frac{9}{48}}$
54. $80^{\frac{1}{2}} + 125^{\frac{1}{2}}$
55. $\sqrt[3]{48} \pm \sqrt[3]{162}$
56. $2\sqrt{48} + 9\sqrt[3]{108}$
57. $3 \times 32^{\frac{1}{4}} + 5 \times 2^{\frac{1}{4}}$
58. $(3a^2b)^{\frac{1}{2}} + (27a^2b)^{\frac{1}{2}}$
59. $(18a^5b^3)^{\frac{1}{2}} + (50a^3b^3)^{\frac{1}{2}}$
60. $\frac{1}{2}\sqrt[3]{\frac{1}{4}} \pm \frac{1}{4}\sqrt[3]{\frac{1}{32}}$
61. $6(4a^2)^{\frac{1}{3}} + 2(2a)^{\frac{1}{3}} + (8a^3)^{\frac{1}{3}}$
62. $\sqrt[3]{192} \pm \sqrt[3]{24}$
63. $(9a^4x)^{\frac{1}{2}} - (4a^4x)^{\frac{1}{2}}$
64. $\frac{1}{2}\sqrt[3]{81} - \frac{1}{4}\sqrt[3]{375}$
65. $b(8a^6b)^{\frac{1}{3}} - a(a^3b^4)^{\frac{1}{3}}$
66. $\frac{1}{3}(486)^{\frac{1}{5}} - \frac{1}{4}(64)^{\frac{1}{5}}$
67. $512^{\frac{1}{3}} - 54^{\frac{1}{3}}$
68. $\sqrt[4]{32} - 2\sqrt[3]{40}$
69. $\sqrt[3]{500} \pm \sqrt[3]{108}$
70. $\sqrt{(45c^3)} + \sqrt{(80c^3)} + \sqrt{(5a^2c)}$.
71. $2\sqrt{\frac{5}{3}} + \frac{1}{6}\sqrt{60} + \sqrt{15} + \sqrt{\frac{3}{5}}$.
72. $7\sqrt[3]{54} + 3\sqrt[3]{16} + \sqrt[3]{2} + \sqrt[3]{(128a^6x^3)}$.
73. $(16a^3b)^{\frac{1}{3}} + (4a^2b)^{\frac{1}{2}} + (54a^3b)^{\frac{1}{3}} + (a^2b)^{\frac{1}{2}}$.
74. $(7\sqrt[3]{54} + 3\sqrt[3]{16} + \sqrt[3]{2}) - 5\sqrt[3]{128}$.
75. $\sqrt[3]{192} + \sqrt[3]{81} - \sqrt[3]{-512}$.
76. $\overline{25a^2b}^{\frac{1}{2}} + \overline{144a^2b}^{\frac{1}{2}} - \overline{289a^2b}^{\frac{1}{2}}$.
77. $\sqrt{\frac{ab^3}{c^2}} + \frac{1}{2c}\sqrt{(a^3b - 4a^2b^2 + 4ab^3)}$.

Multiply

78. $3\sqrt{b}$ by $5\sqrt{x}$
79. $4^{\frac{1}{10}}$ by $5^{\frac{1}{10}}$
80. $2\sqrt[3]{3}$ by $4\sqrt[3]{72}$
81. $a^{\frac{1}{2}}$ by $a^{\frac{1}{3}}$
82. $3b^{\frac{1}{2}}$ by $4a^{\frac{1}{4}}$
83. $\frac{1}{2}\sqrt[3]{\frac{2}{3}}$ by $\frac{3}{4}\sqrt[3]{\frac{5}{6}}$
84. $2^{\frac{1}{2}}$ by $3^{\frac{1}{3}}$ by $5^{\frac{1}{4}}$
85. $3\sqrt{(4+6\sqrt{2})}$ by $5\sqrt{2}$
86. $x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}$ by $(a+x)^{\frac{1}{2}}$
87. $(a+x)^{\frac{2}{3}}$ by $(a-x)^{\frac{2}{3}}$
88. $\frac{6^{\frac{1}{2}}}{4}$ by $2\,\frac{9^{\frac{1}{3}}}{15}$
89. $5\sqrt{3}$ by $7\sqrt{\frac{8}{3}}$ by $\sqrt{2}$.
90. $\sqrt[3]{4}$ by $7\sqrt[3]{6}$ by $\frac{1}{2}\sqrt[3]{5}$
91. $a^{\frac{1}{2}}$ by $x^{\frac{1}{3}}$
92. 4 by $2\sqrt[6]{3}$ by $\sqrt[6]{72}$
93. $a^{\frac{1}{3}}$ by $ba^{\frac{1}{3}}$ by $ca^{\frac{1}{3}}$
94. $(4 + 2\sqrt{2})$ by $(2 - \sqrt{2})$
95. $\frac{1}{2}\sqrt[3]{18}$ by $5\sqrt[3]{20}$
96. $(1+x)^{\frac{1}{2}}$ by $(1-x)^{\frac{1}{2}}$
97. $x^{\frac{m}{n}}$ by $x^{\frac{p}{q}}$
98. $(a + b^{\frac{1}{2}})^{\frac{1}{2}}$ by $(c + d^{\frac{1}{2}})^{\frac{1}{2}}$

99. $5\sqrt[3]{4} - 2\sqrt[3]{16}$ by $2\sqrt[3]{2} - 3\sqrt[3]{4}$.

100. $x + y^{\frac{1}{2}} + z^{\frac{1}{2}}$ by $x - y^{\frac{1}{2}} + z^{\frac{1}{2}}$.

101. $a^{\frac{3}{2}} - ax^{\frac{1}{2}} + a^{\frac{1}{2}}x - x^{\frac{3}{2}}$ by $a + a^{\frac{1}{2}}x^{\frac{1}{2}} + x$.

102. $\sqrt{a} + \sqrt{b} + \sqrt{c}$ by $\sqrt{a} + \sqrt{b} - \sqrt{c}$ by $\sqrt{a} + \sqrt{c} - \sqrt{b}$ by $\sqrt{b} + \sqrt{c} - \sqrt{a}$.

103. $(a^{2m-n}b^{5m+1}c^{3p})^{\frac{1}{m}}$ by $(a^{n}b^{m-1}\,c^{m-3p})^{\frac{1}{m}}$.

104. $x + \frac{p}{2} + \sqrt{q + \frac{p^2}{4}}$ by $x + \frac{p}{2} - \sqrt{q + \frac{p^2}{4}}$.

105. $3 + \sqrt{5}$ by $2 - \sqrt{5}$.

106. $7 + 2\sqrt{6}$ by $9 - 5\sqrt{6}$.

107. $7\sqrt{6} + 2\sqrt{3}$ by $\sqrt{5} + \sqrt{6}$.

108. $\sqrt[3]{(a^2 - x^2)^2}$ by $\sqrt[3]{(a^2 - x^2)}$.

109. $2\sqrt{8} + 3\sqrt{5} - 7\sqrt{2}$ by $\sqrt{72} - 5\sqrt{20} - 2\sqrt{2}$.

110. $\sqrt[n]{(a + \sqrt{b})}$ by $\sqrt[n]{(a - \sqrt{b})}$.

Divide

111. $\sqrt{40}$ by $\sqrt{2}$
112. $6\sqrt{54}$ by $3\sqrt{2}$
113. $\sqrt[3]{135}$ by $\sqrt[3]{5}$
114. $\sqrt[3]{4}$ by $\frac{1}{2}\sqrt[3]{6}$
115. $72^{\frac{1}{6}}$ by $3^{\frac{1}{2}}$
116. $x^{\frac{4}{3}}$ by x
117. $\sqrt{48}$ by 2
118. $(1 - x^2)^{\frac{1}{2}}$ by $(1 + x)^{\frac{1}{2}}$
119. $(n^2 - m^2)^{\frac{1}{2}}$ by $n - m$
120. $(ab^2 - b^3x)^{\frac{1}{2}}$ by $(a - x)^{\frac{1}{2}}$
121. $\sqrt{72} + \sqrt{32} - 4$ by $\sqrt{8}$
122. $a^{\frac{1}{3}}$ by $a^{\frac{1}{6}}$
123. $2\sqrt[3]{\frac{2}{3}}$ by $3\sqrt[3]{\frac{3}{4}}$
124. $5\frac{3}{4}\sqrt{\frac{1}{135}}$ by $\frac{2}{3}\sqrt{\frac{1}{5}}$
125. $a^{\frac{1}{2}}$ by $a^{\frac{1}{3}}$
126. $64^{\frac{1}{3}}$ by 2
127. $4\sqrt[3]{12}$ by $2\sqrt{3}$
128. $10\sqrt[3]{108}$ by $5\sqrt[3]{4}$
129. $a^{\frac{5}{2}} - x^{\frac{5}{2}}$ by $a^{\frac{1}{2}} - x^{\frac{1}{2}}$
130. $(\frac{a}{b})^{\frac{1}{4}}$ by $(\frac{a}{b})^{\frac{1}{3}}$
131. $\frac{5}{2}x^{\frac{1}{n}}$ by $\frac{7}{2}x^{\frac{1}{m}}$
132. $x^{\frac{1}{4}}$ by $x^{\frac{1}{3}}$
133. $10\sqrt{27}$ by $2\sqrt{3}$
134. $x^{\frac{2}{3}}$ by x
135. $x^{\frac{m}{n}}$ by x
136. 4 by $\sqrt{8}$

137. $(a^3 - a^2(a^3b^2)^{\frac{1}{3}} - 2(a^2b^3)^{\frac{1}{2}} + 2b^{1\frac{1}{2}})$ by $(a^{\frac{1}{3}} - b^{\frac{1}{3}})$.

138. $a\sqrt{x} + \sqrt{bx} - a\sqrt{y} - \sqrt{by}$ by $\sqrt{x} - \sqrt{y}$.

139. $\sqrt{6} + 4\sqrt{18} - 3 - 8\sqrt{6}$ by $\sqrt{3}$.

140. $4a^{\frac{8}{3}} - 9a^{\frac{7}{3}} + 14a - 19a^{\frac{2}{3}} + 4a^{\frac{1}{3}}$ by $a^{\frac{5}{3}} - 2a^{\frac{4}{3}} + 3a^{\frac{2}{3}} - 4$.

Extract the roots indicated,—

141. Square root of $7 + 4\sqrt{3}$.
142. „ „ $6 - 2\sqrt{5}$.
143. „ „ $7 - 2\sqrt{10}$.
144. „ „ $2 + \sqrt{3}$.
145. „ „ $\frac{3}{2} + \sqrt{2}$.
146. „ „ $23 + 8\sqrt{7}$.
147. „ „ $\sqrt{32} - \sqrt{24}$.
148. „ „ $11 + \sqrt{72}$.
149. „ „ $94 - 42\sqrt{5}$.
150. „ „ $3\sqrt{5} + \sqrt{40}$.
151. „ „ $\sqrt{27} + 2\sqrt{6}$.
152. „ „ $4\sqrt{3} + 6$.
153. „ „ $8 + 2\sqrt{2} + 2\sqrt{5} + 2\sqrt{10}$.
154. Cube root of $10 + \sqrt{108}$.
155. „ „ $16 + 8\sqrt{5}$.

Convert into rational quantities,

156. $a + \sqrt{b}$
157. $-5 - \sqrt{\frac{3}{4}}$
158. $\sqrt{a} + \sqrt{b}$
159. $a\sqrt{b} + c\sqrt{d}$
160. $9 + 2\sqrt{10}$

Convert into fractions with rational denominators,

161. $\dfrac{1}{\sqrt{3} + 2}$
162. $\dfrac{1 + \sqrt{2}}{2 - \sqrt{2}}$
163. $\dfrac{\sqrt{20} + \sqrt{12}}{\sqrt{5} - \sqrt{3}}$
164. $\dfrac{\sqrt{a}}{b + \sqrt{c}}$
165. $\dfrac{\frac{1}{2}\sqrt{\frac{1}{2}}}{\sqrt{2} + 3\sqrt{\frac{1}{2}}}$
166. $\sqrt[n]{\dfrac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}}}$
167. $\dfrac{3}{\sqrt[3]{5} - \sqrt[3]{2}}$
168. $\dfrac{\sqrt{(1 - x)} + \dfrac{1}{\sqrt{(1 + x)}}}{1 + \dfrac{1}{\sqrt{(1 - x^2)}}}$
169. $\dfrac{\sqrt{(x + a)} + \sqrt{(x - a)}}{\sqrt{(x + a)} - \sqrt{(x - a)}}$

170. Shew that $\dfrac{a\sqrt{(a + x)}}{\sqrt{(a + x)} - \sqrt{x}} = a + x + \sqrt{(dx + x^2)}$.

XI. IMAGINARY QUANTITIES.

118. The even roots of negative quantities, as has been already remarked in Art. 96, are not assignable by any process of Evolution; and, hence, expressions involving such roots have been called *imaginary* or *impossible* quantities. In certain cases, however, a real value can be assigned to them, but not by ordinary Evolution.

119. Since (Art. 95) the square root of a product equals the product of the square roots of its factors, and since $-a^2 = a^2 \times -1$, it follows that $\sqrt{-a^2} = \sqrt{a^2} \times \sqrt{-1} = a\sqrt{-1}$; so also $\sqrt{-b} = \sqrt{b} \times \sqrt{-1}$ and $\sqrt[4]{-a^4} = a\sqrt[4]{-1}$, &c.; and thus any imaginary quantity may be resolved into two factors, one a real quantity, and the other the imaginary expression $\sqrt{-1}$. Further, from the nature of powers and roots it is plain that the square of $\sqrt{-1}$ is -1, or that $\sqrt{-1} \times \sqrt{-1} = -1$. Attention to this principle, and to the above mode of notation, will render all the algebraic operations with imaginary quantities very easily performed.

Thus, the sum of $a + 3b\sqrt{-1}$ and $a - 2b\sqrt{-1}$ is $2a + b\sqrt{-1}$; and the product of $a\sqrt{-b}$ by $c\sqrt{-d}$ is $a\sqrt{b}\sqrt{-1} \times c\sqrt{d}\sqrt{-1} = ac\sqrt{bd} \times -1 = -ac\sqrt{bd}$. Also to divide $a\sqrt{-1}$ by $b\sqrt{-1}$, we have $\frac{a\sqrt{-1}}{b\sqrt{-1}} = \frac{a}{b}$. These two latter results are real, the imaginary quantities having disappeared. The same occurs frequently in operating with these quantities; as in the following example, which is besides otherwise interesting.

Let it be required to multiply $a + b\sqrt{-1}$ by $a - b\sqrt{-1}$. The work is as follows :—

$$\begin{array}{l}
a + b\sqrt{-1} \\
\underline{a - b\sqrt{-1}} \\
a^2 + ab\sqrt{-1} \\
\underline{\quad - ab\sqrt{-1} + b^2} \\
\quad a^2 + b^2
\end{array}$$

Hence, conversely, it appears that we can resolve a binomial whose terms are positive into two compound factors, which are the sum and difference of a real and an imaginary quantity. Thus, since $16a^2 + 9x^2 = 16a^2 - 9x^2 \times -1 = 9x^2 - 16a^2 \times -1$; and since the square roots of the terms are $4a$ and $3x\sqrt{-1}$, the factors of the given binomial will be $4a + 3x\sqrt{-1}$, and $4a - 3x\sqrt{-1}$, or $3x + 4a\sqrt{-1}$, and $3x - 4a\sqrt{-1}$. Such expressions as these, which differ only in the sign of the coefficient of $\sqrt{(-1)}$, are called *conjugate.*

120. In order to determine the sign of the product of any number of imaginary quantities it is not sufficient that we know merely the signs which are prefixed to these quantities; it is essential that we know also how often $\sqrt{-1}$ is involved; and hence it is of importance to determine the successive powers of this expression. Now $(\sqrt{-1})^0 = 1$ (Art. 84), $(\sqrt{-1})^1 = \sqrt{-1}$, $(\sqrt{-1})^2 = -1$; therefore $(\sqrt{-1})^3 = -1 \times \sqrt{-1} = -\sqrt{-1}$, $(\sqrt{-1})^4 = -1 \times -1 = 1$; and in the same manner we have,

1. $(\sqrt{-1})^{4n} = [(\sqrt{-1})^4]^n = 1^n = 1.$
2. $(\sqrt{-1})^{4n+1} = (\sqrt{-1})^{4n} \times \sqrt{-1} = \sqrt{-1}.$
3. $(\sqrt{-1})^{4n+2} = (\sqrt{-1})^{4n} \times (\sqrt{-1})^2 = -1.$
4. $(\sqrt{-1})^{4n+3} = (\sqrt{-1})^{4n+2} \times \sqrt{-1} = -\sqrt{-1}.$

It is plain that all the powers of $\sqrt{-1}$ are comprehended in these four forms, since n being a whole number or zero, all positive integers are of one or other of the forms $4n$, $4n + 1$, $4n + 2$, $4n + 3$. Thus,

if $n = 0$ these forms are the 0th, 1st, 2d, 3d, powers of $\sqrt{-1}$,
„ $n = 1$ „ 4th, 5th, 6th, 7th „
„ $n = 2$ „ 8th, 9th, 10th, 11th, „
„ $n = 3$ „ 12th, 13th, 14th, 15th, &c.

Hence,

the 0th, 4th, 8th, 12th, $4n$th powers of $\sqrt{-1}$ are each 1,
„ 1st, 5th, 9th, 13th $(4n + 1)$th „ $\sqrt{-1}$,
„ 2nd, 6th, 10th, 14th $(4n + 2)$th „ -1,
„ 3rd, 7th, 11th, 15th $(4n + 3)$th „ $-\sqrt{-1}$,
and so on of the others.*

EXERCISES.

1. Find the sum and difference of $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$.
2. Add together $a + b\sqrt{-1}$ and $c + d\sqrt{-1}$.
3. From $4a + 2b\sqrt{-1}$ subtract $3a + 4b\sqrt{-1}$.
4. Multiply $\sqrt{-a^2}$ by $\sqrt{-b^2}$.
5. Multiply $\sqrt{-x}$ by $\sqrt{-y}$.
6. Multiply $a\sqrt{-1}$ by $-b\sqrt{-1}$.
7. Find the 3rd and 4th powers of $a\sqrt{-1}$.

---

* The same may, perhaps, be more easily found by actual multiplication, after the first four powers are known. Thus, the fifth power is $\sqrt{-1}$, being the product of the 4th, 1, by the 1st, $\sqrt{-1}$; the 6th is -1, the product of the 4th, 1, by the 2nd, -1; also multiplying the 3d power, $-\sqrt{-1}$, by the 4th power, 1, we find for the 7th $-\sqrt{-1}$; multiplying the 4th power by itself we find 1 for the 8th; the 5th power $\sqrt{-1}$, multiplied by the 4th produces $\sqrt{-1}$ for the 9th, and so on for the 10th, 11th, and all higher powers.

8. Resolve 2 or $1 + 1$ into imaginary factors.

9. Multiply $2\sqrt{-3}$ by $3\sqrt{-2}$.

10. Find the 3d power of $a + b\sqrt{-1}$.

11. Multiply $a \pm b\sqrt{-1}$ by $c \pm d\sqrt{-1}$.

12. Resolve $25a^4 + 16b^6$ into imaginary factors. *See Note, p.* 32.

13. Find the sum and difference of the fractions

$$\frac{1}{a-b\sqrt{-1}},\ \frac{1}{a+b\sqrt{-1}};\ \text{and also of}$$

$$\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}},\ \frac{a-b\sqrt{-1}}{a+b\sqrt{-1}}.$$

14. Resolve $a + b$ into imaginary factors.

15. Find the cube of $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$ and $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$.

16. Divide $6\sqrt{-3}$ by $2\sqrt{-4}$.

17. Simplify the fraction $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}$.

18. Resolve 100 into imaginary factors.

19. Find the continued product of $x + a$, $x + a\sqrt{-1}$, $x - a$, and $x - a\sqrt{-1}$.

20. Shew that $\dfrac{a+b\sqrt{-1}}{a-b\sqrt{-1}} = \dfrac{a^2-b^2+2ab\sqrt{-1}}{a^2+b^2}$.

21. Of what number are $24 + 7\sqrt{-1}$, and $24 - 7\sqrt{-1}$ the imaginary factors?

22. Divide $3 - \sqrt{-2}$ by $2 + 3\sqrt{-2}$.

23. Multiply $e^{a\sqrt{-1}}$ by $e^{b\sqrt{-1}}$.

24. Raise $e^{a\sqrt{-1}}$ to the *n*th power.

25. Of what number are $20 + 15\sqrt{-1}$ and $20 - 15\sqrt{-1}$ the imaginary factors?

XII. SIMPLE EQUATIONS

WITH ONE UNKNOWN QUANTITY.

121. A statement of equality between two expressions, is called an *equation;* and the expressions are called the *members* or *sides* of the equation; thus, $4b + d = 2a - c$ is an equation, and its sides or members are $4b + d$ and $2a - c$.

122. The chief use of an equation is to enable us to express one quantity in terms of others. Hence, if one of the quantities in an equation be unknown, and the others known, by expressing the former in terms of the latter, its value is determined. It is customary to denote unknown quantities by the last letters of the alphabet; while known quantities, if not expressed in numbers, are denoted by the first letters of the alphabet.

123. A *simple* equation, or an equation of the *first degree,* is one which contains the first power only of the unknown quantity.

124. *Any term may be transposed from one side of an equation to the other, if its sign be changed at the same time.*

Let $x + a = b$, then $x = b - a$; for this is nothing else than subtracting a from both sides (Art. 28).

Again, let $x - a = b$, then $x = b + a$; for this is nothing else than adding a to both sides (Art. 27).

125. *An equation may be cleared of fractions, by multiplying all the terms by the least common multiple of the denominators* (Art. 63).*

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* This may also be effected, but not so readily, by multiplying all the terms either by each denominator in succession, by the product of all the denominators, or by any common multiple of them.

For, by this means, each numerator is rendered divisible by the denominator, (Arts. 46, 48, 54), and (Art. 29) the two sides are still equal to one another. (*See* Example 4, Art. 76). Thus, to free the equation, $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 1$, from fractions, multiply each term by 12, the least common multiple of the denominators; then we have $6x + 4x + 3x = 12$, in which the terms are integral.

126. *An equation may be cleared of surds, by transposing* (Art. 124) *the terms, so that each surd, in succession, may occupy one side, and the remaining terms the other side, and then raising each side to a power corresponding to the index of the surd.*

For (Arts. 87, 31) the surds will be made to disappear successively, and each result will still be an equation. Thus, let $\sqrt[n]{(a + x)} + b = c$;
then, by transposition (Art. 124), $\sqrt[n]{(a + x)} = c - b$;
and, by involution, $a + x = (c - b)^n$,
which is free from irrational terms. If there are more surds than one, the process must be repeated.

127. To solve a simple equation. (1) *Clear the equation of fractions and surds;*[*] (2) *transpose the known terms to one side, and the unknown terms to the other, incorporating like quantities;* (3) *and divide both members* (Art. 30) *by the coefficient of the unknown quantity.*

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[*] That is, of course, when the unknown quantity is contained in the fraction or surd. Before commencing the solution, we may sometimes simplify an equation, by transposing and incorporating like terms, or by division (Art. 30).

EXAMPLES.

1. Given $4 - 9x = 14 - 11x$ to find the value of x.

Transposing (part 2 of the Rule, and Art. 124), we have

$$11x - 9x = 14 - 4;$$

and collecting the terms, $2x = 10$;
therefore, by division (part 3), $x = 5$, the required value.

To verify this value of x, put 5 for x in the given equation, and it becomes $4 - 45 = 14 - 55$;
that is, $-41 = -41$;
or, $41 = 41$,
by changing both signs (Art. 124), or dividing both sides by -1.

When the two sides thus become identical, the value found for the unknown is said to be that which *satisfies* the equation, or for which the equation *holds true*.*

2. Given, $3x^2 + 4x - \frac{x^2}{3} - 46x + 2x^2 = 0$, to find the value of x.

Here, simplifying before clearing of fractions (Note to the Rule), and multiplying both terms by 3, we easily find $x = 9$.

3. Given, $\frac{2y+3}{2y-3} = \frac{11}{9}$, required y.

This equation may be easily solved, by taking the sum and difference of the terms, and dividing the former of these by the latter. We thus find $\frac{4y}{6} = \frac{20}{2}$, and therefore $y = 15$.

---

* It is of great use to employ this test frequently, in the solution of equations. Thus, in the above example, if we had forgotten to change the sign of 4, in transposing it to the right side, we should have had $11x - 9x = 18$, or $2x = 18$, and $x = 9$; and, in attempting to verify this, we should have found $4 - 81 = 14 - 99$, or $77 = 85$, an impossibility; which proves that 9 is not the true value of x.

4. Given, $\frac{5x+4}{2} : \frac{18-x}{4} :: 7 : 4$, to find x.

Since, in a proportion, the product of the means is equal to the product of the extremes (Euc. VI. 16), it follows that

$$\frac{5x+4}{2} \times 4 = \frac{18-x}{4} \times 7,$$

whence, we easily find $x = 2$.

5. Given, $\frac{4x+3}{9} + \frac{7x-29}{5x-12} = \frac{8x+19}{18}$; required x.

Here we multiply both sides by 18, cancel $8x$, transpose 6, multiply by $5x-12$, transpose and divide, and at last we find $x = 6$.

6. Given, $\frac{3x-1}{2} - \frac{2x-4}{3} = 10$; find x.

Here, in clearing of fractions it is to be carefully observed that the signs of the third and fourth terms are both changed in virtue of the *minus* preceding; we thus get $9x - 3 - 4x + 8 = 60$, and $x = 11$.

7. Given, $3{\cdot}25x - 5{\cdot}007 - x = {\cdot}2 - {\cdot}34x$.
By transposition, we have

$$3{\cdot}25x - x + {\cdot}34x = 5{\cdot}207\,;$$

that is, $\quad x(3{\cdot}25 - 1 + {\cdot}34) = 5{\cdot}207$;

whence, $\quad 2{\cdot}59x = 5{\cdot}207$

and $\quad x = \frac{5{\cdot}207}{2{\cdot}59} = 2{\cdot}0104\ldots$

8. Given, $ax - \frac{a^2-3bx}{a} - ab^2 = bx + \frac{6bx-5a^2}{2a} - \frac{bx+4a}{4}$

Here, $4a$ is the L. C. M. of the denominators: multiplying by it, transposing and adding,
we obtain $\quad x(4a^2 - 3ab) = 4a^2b^2 - 10a^2$;

and by division and reduction $x = \dfrac{4ab^2 - 10a}{4a - 3b}$.

9. Given, $\dfrac{ax - b^2}{\sqrt{ax} + b} = c + \dfrac{\sqrt{ax} - b}{c}$; required the value of x.

In this case, before squaring, the equation may be simplified. The numerator of the first term is divisible by the denominator. Performing the division there is obtained

$$\sqrt{ax} - b = c + \frac{\sqrt{ax} - b}{c}.$$

By transposition and reduction to a common denominator this becomes $\dfrac{c(\sqrt{ax} - b)}{c} - \dfrac{\sqrt{ax} - b}{c} = c$;

whence, separating the factors, $\dfrac{c - 1}{c}(\sqrt{ax} - b) = c$.

Dividing by the known coefficient, and then transposing b, we obtain $\sqrt{ax} = b + \dfrac{c^2}{c - 1}$;

squaring both sides, and dividing by a, there results

$$x = \frac{1}{a}\left(b + \frac{c^2}{c - 1}\right)^2.$$

10. Given $\sqrt{a} + \sqrt{x} = \sqrt{ax}$, to find the value of x.

Transposing $\sqrt{x}$, and resolving the right member into its factors, we have $\sqrt{a} = \sqrt{x}(\sqrt{a} - 1)$; whence, by division and involution, $x = \dfrac{a}{(\sqrt{a} - 1)^2}$.

11. Given, $\dfrac{\sqrt{a} + \sqrt{(a - x)}}{\sqrt{a} - \sqrt{(a - x)}} = \dfrac{1}{a}$, required x.

Employing the same artifice as is used in Ex. 3, we shall have $\sqrt{\dfrac{a}{a - x}} = \dfrac{1 + a}{1 - a}$.

Squaring this, inverting the terms, and transposing, we obtain
$$\frac{x}{a} = 1 - \frac{1 - 2a + a^2}{1 + 2a + a^2} = \frac{4a}{(1 + a)^2};$$
hence, $x = \dfrac{4a^2}{(1 + a)^2} = \left(\dfrac{2a}{1 + a}\right)^2.$

The solution will not, in general, be simplified by the use of this artifice, unless when the unknown occurs only on one side.

12. Given, $\dfrac{1}{x} + \dfrac{1}{a} = \sqrt{\dfrac{1}{a^2} + \sqrt{\left(\dfrac{4}{a^2x^2} + \dfrac{9}{x^4}\right)}}$ to find the value of x.

Squaring both sides, and cancelling equal terms, we find
$$\frac{1}{x^2} + \frac{2}{ax} = \sqrt{\left(\frac{4}{a^2x^2} + \frac{9}{x^4}\right)}.$$

Multiply the left member by x, and the right by its equal, $\sqrt{x^2}$, and there will be obtained,
$$\frac{1}{x} + \frac{2}{a} = \sqrt{\left(\frac{4}{a^2} + \frac{9}{x^2}\right)};$$
squaring, $\dfrac{1}{x^2} + \dfrac{4}{ax} + \dfrac{4}{a^2} = \dfrac{4}{a^2} + \dfrac{9}{x^2}.$

Hence, $\dfrac{1}{a} = \dfrac{2}{x}$ and $x = 2a.$

13. Given, $\sqrt[3]{(a + x)} + \sqrt[3]{(a - x)} = b$; required x.

Since $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$ (Art. 87), we have in this case,
$$a + x + a - x + 3\sqrt[3]{(a^2 - x^2)}.(\sqrt[3]{a + x} + \sqrt[3]{a - x}) = b^3;$$
but $\sqrt[3]{a + x} + \sqrt[3]{a - x} = b,$

therefore $2a + 3b\sqrt[3]{(a^2 - x^2)} = b^3,$

and $3b\sqrt[3]{(a^2 - x^2)} = b^3 - 2a;$

by division, involution, and transposition,

$$x^2 = a^2 - \left(\frac{b^3 - 2a}{3b}\right)^2;$$

therefore, $$x = \sqrt{a^2 - \left(\frac{b^3 - 2a}{3b}\right)^2};$$

14. Let $(3 + y)^{\frac{1}{n}} = (10 + y^2)^{\frac{1}{2n}}$, to find y.

Raising both members to the $2n$th power will give

$$(3 + y)^2 = 10 + y^2;$$

whence, $$y = \tfrac{1}{6}.$$

15. Given $(a^2 + c)^{\frac{1}{3}} = \left(\frac{a^2 + c}{d(x + b)}\right)^{\frac{1}{4}}$; required x.

Involving both members to the 4th power, and clearing of fractions, we find $(a^2 + c)^{\frac{4}{3}} . d(x + b) = a^2 + c$.

Dividing both terms by $a^2 + c$, that is, by $\sqrt[3]{(a^2 + c)^3}$, we find $$\sqrt[3]{(a^2 + c)} . d(x + b) = 1:$$

hence, $$x = \frac{1}{d\sqrt[3]{(a^2 + c)}} - b.$$

EXERCISES.

I. *Numerical Equations.*

1. $x + 4 = 7$
2. $x - 6 = 1$
3. $2 + x - 3 + 7 = 10 - 3$
4. $x - 2 = \pm 8$.
5. $y - \frac{1}{2} = \frac{1}{4}$
6. $3y + 4 = 2y + 10$
7. $5y - 15 = 2y - 6$
8. $x + 15 = 35 - 3x$
9. $19x + 13 = 59 - 4x$
10. $z - 7 = \pm 5$
11. $z + 4 = \pm 10$
12. $4v + 26 = 59 - 7v$
13. $11v - 14 = 9v - 8$
14. $8x + 12 = 5x + 30$
15. $x + 18 = 3x - 5$
16. $2x - 1 = 5x - 19$
17. $5x + 12 = 2x + 21$
18. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 13$

19. $\frac{x}{2}-\frac{x}{3}-\frac{x}{4}+\frac{x}{6}=1$

20. $\frac{x}{5}+\frac{x}{4}+\frac{x}{3}-\frac{x}{2}=17$

21. $\frac{2x}{5}-\frac{x}{6}+\frac{x}{2}-44=0$

22. $\frac{x}{3}+6x=\frac{4x-2}{5}$

23. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=77$

24. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=14$

25. $0=11-\frac{8x}{5}-4+\frac{9x}{10}$

26. $\frac{5x-7}{2}-\frac{2x+7}{3}=3x-14$

27. $\frac{x}{2}-\frac{x}{3}-\frac{x}{4}+\frac{4}{3}=\frac{3}{4}$

28. $\frac{x+4}{3}+\frac{x-2}{2}=7$

29. $\frac{10+x}{5}:\frac{4x-9}{7}=14:5$

30. $\frac{x+1}{2}-\frac{x+4}{3}=1$

31. $\frac{2x+2}{3}-\frac{x+4}{6}+8=5$

32. $\frac{x+3}{2}+\frac{x}{3}=4-\frac{x-5}{4}$

33. $\frac{3}{x}-\frac{2}{x+1}=\frac{5}{4(x+1)}$

34. $\frac{6x+7}{15}-\frac{2x-2}{7x-6}=\frac{2x+1}{5}$

35. $\frac{6x+13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5}$

36. $\frac{9x+20}{36}-\frac{4x-12}{5x-4}=\frac{x}{4}$

37. $\frac{2}{2x-5}+\frac{1}{x-3}=\frac{6}{3x-1}$

38. $4+x-5-2x+2=5-2x+6.$

39. $3x+6-x+3=16+x-4.$

40. $7-3x+8+x=12-3x+8.$

41. $-(8+11x)=-(18+13x).$

42. $-(10-9x)=-(16-7x).$

43. $5(5x-6)-4(4x-5)+3(3x-2)-2x-16=0.$

44. $10\left(x+\frac{1}{2}\right)-6x\left(\frac{1}{x}-\frac{1}{3}\right)=23.$

45. $3x+\frac{2x+6}{5}=5+\frac{11x-37}{2}$

46. $\frac{12x-2}{6}-\frac{18-4x}{3}=x+2.$

47. $\frac{1}{7}\left(x-\frac{1}{2}\right)-\frac{1}{5}\left(\frac{2}{3}-x\right)=1\frac{13}{30}.$

48. $\frac{x-4\frac{2}{3}}{3}-\frac{2x-3\frac{2}{3}}{4}=\frac{3}{2}\left\{x-\frac{x-1\frac{1}{2}}{2}\right\}+\frac{4x}{3}\left\{x-3-\right.$
$\left.\frac{(x-1)(x-2)}{x}\right\}.$

49. $\frac{7x+5}{23}+\frac{9x-1}{10}-\frac{x-9}{5}+\frac{2x-3}{15}=23\frac{1}{3}.$

50. $\frac{25-\frac{1}{3}x}{x+1}+\frac{16x+4\frac{1}{5}}{3x+2}=\frac{23}{x+1}+5.$

51. $\frac{x}{2}-\frac{\frac{1}{3}(2x-3)-\frac{1}{4}(3x-1)}{\frac{1}{2}(x-1)}=\frac{3}{2}\cdot\frac{x^2+2}{3x-2}.$

52. $\frac{3x-\frac{2}{3}(1+x)}{4}+\frac{1-\frac{1}{5}x}{5\frac{1}{2}}=\frac{2\frac{2}{5}+\frac{1}{25}(x-1)}{2\frac{1}{5}}.$

53. $\frac{2x+3\frac{1}{2}}{9}-\frac{13x-22}{17x-32}+\frac{x}{3}=\frac{7x}{12}-\frac{x+16}{36}.$

54. $\frac{4}{x+2}+\frac{7}{x+3}=\frac{37}{(x+2)(x+3)}.$

55. $\frac{18x-19}{28}+\frac{11x+21}{6x+14}=\frac{9x+15}{14}.$

56. $\frac{8x+5}{14}+\frac{7x-3}{6x+2}=\frac{16x+15}{28}+\frac{2\frac{1}{4}}{7}.$

57. $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}.$

58. $\cdot 15x+\cdot 2-\cdot 875x+\cdot 375=\cdot 0625x-1.$

59. $\frac{9x-8}{7}+\frac{x+\cdot 2}{\cdot 4}+\frac{2}{9}=0.$

60. $1\cdot 2x-\frac{\cdot 18x-\cdot 05}{\cdot 5}=\cdot 4x+8\cdot 9.$

II. *Literal Equations.*

61. $bx=a$

62. $\frac{y}{b}=4d$

63. $ay-b=by-c$

64. $bx-6a=3c-2a-4x$

65. $dx-5ac=3bc-x$

66. $\frac{1}{a}+\frac{b}{z}=c$

67. $\dfrac{3a+x}{x} - 5 - \dfrac{6}{x} = 0.$

68. $6b + ax - 1 = 6 + 7x + 3b$

69. $x + \dfrac{n}{m}x + \dfrac{p}{m}x = r$

70. $\dfrac{a}{1+x} + \dfrac{a}{1-x} = b$

71. $\dfrac{x}{a} + \dfrac{x}{b} + \dfrac{x}{c} = d$

72. $\dfrac{ax}{b} + \dfrac{cx}{d} + \dfrac{ex}{f} - g = h$

73. $a - bx^{-2} = b + ax^{-2}$

74. $\dfrac{a}{b+x} + \dfrac{a}{b-x} = c$

75. $4b - 3dy + 7 = 4dy - ay + 6 + 5b.$

76. $\frac{5}{6}ab + \frac{4}{5}ac - \frac{2}{3}cx = \frac{3}{4}ac + 2ab - 6cx.$

77. $\dfrac{1}{2}\left(x - \dfrac{a}{3}\right) - \dfrac{1}{3}\left(x - \dfrac{a}{4}\right) + \dfrac{1}{4}\left(x - \dfrac{a}{5}\right) = 0.$

78. $\dfrac{ax-b}{4} + \dfrac{a}{3} = \dfrac{bx}{2} - \dfrac{bx-a}{3}.$

79. $(a+x)(b+x) - a(b+c) = \dfrac{a^2c}{b} + x^2.$

80. $\dfrac{x-a}{b} + \dfrac{x-b}{c} + \dfrac{x-c}{a} = \dfrac{x-(a+b+c)}{abc}.$

81. $6x + a : 4x + b = 3x - b : 2x - a.$

82. $x + 1 : x - 1 :: a + 2b : a - 3b.$

83. $\dfrac{3abc}{a+b} + \dfrac{a^2b^2}{(a+b)^3} + \dfrac{(2a+b)b^2x}{a(a+b)^2} = 3cx + \dfrac{bx}{a}.$

84. $\dfrac{a}{bx} + \dfrac{c}{dx} + \dfrac{e}{fx} + \dfrac{g}{hx} - k = 0.$

III. *Equations with surds.*

85. $\sqrt{x} = a$

86. $\sqrt{(x+6)} = 3$

87. $\sqrt{x} - 7 = -3$

88. $\sqrt[3]{y} + 5 = 11$

89. $\sqrt[3]{(y+4)} = 5$

90. $(3y-5)^{\frac{1}{2}} = (7-y)^{\frac{1}{2}}$

91. $\sqrt{(x+a)} = \sqrt{x} + a$

92. $\sqrt{(x+5)} + 3 = 8 - \sqrt{x}$

93. $\sqrt{1 + \sqrt{(3 + \sqrt{6x})}} = 2$

94. $(12+x)^{\frac{1}{2}} = 2 + x^{\frac{1}{2}}$

95. $\sqrt[4]{(11 + \sqrt{5x})} - 3 = 0$

96. $\sqrt{(x-a)} = \sqrt{x} - \frac{1}{2}\sqrt{a}$

97. $\dfrac{x - ax}{\sqrt{x}} = \dfrac{\sqrt{x}}{x}$

98. $\sqrt{(a+x)} + \sqrt{(a-x)} = b$

99. $(7+x)^{\frac{1}{2}} + x^{\frac{1}{2}} = \dfrac{8}{(7+x)^{\frac{1}{2}}}$

100. $\sqrt{(a+x)} + \sqrt{\dfrac{a}{x}} = \sqrt{x}$

101. $(x^2 - b^2)^{\frac{1}{4}} = (a-x)^{\frac{1}{2}}$

102. $\dfrac{4a+\sqrt{x}}{3b+\sqrt{x}} = \dfrac{2a+\sqrt{x}}{b+\sqrt{x}}$

103. $\dfrac{\sqrt{x}-b}{\sqrt{ax}+3b} = \dfrac{\sqrt{ax}-2b}{\sqrt{ax}+5b}$

104. $\dfrac{x^{\frac{1}{2}}+b^{\frac{1}{2}}}{x^{\frac{1}{2}}-b^{\frac{1}{2}}} = \dfrac{a}{b}$

105. $\sqrt[m]{(ax+b)} = \sqrt[n]{(cx+d)}$.

106. $\sqrt{(x+2)} \times \sqrt{5} = \sqrt{5x}+2$.

107. $\sqrt{(2+x)} + \sqrt{x} = \dfrac{4}{\sqrt{(2+x)}}$.

108. $\sqrt[4]{\{-1+\sqrt[3]{(\sqrt{7x}+2)}\}} = 1$.

109. $\sqrt{\dfrac{a^2-x^2}{a-x}} + x - a = 2x$.

110. $\sqrt{[11+\sqrt[3]{-13-\sqrt{20+\sqrt[4]{(3x+25)}}}]} = 3$.

111. $(x+x^{\frac{1}{2}})^{\frac{1}{2}} - (x-x^{\frac{1}{2}})^{\frac{1}{2}} = \frac{3}{2}\left(\dfrac{x}{x+x^{\frac{1}{2}}}\right)^{\frac{1}{2}}$.

112. $\dfrac{3x-16}{\sqrt{3x}+4} = a + \dfrac{\sqrt{3x}-4}{a}$.

113. $(a+x)^{\frac{2}{3}} + (a-x)^{\frac{2}{3}} = 3(a^2-x^2)^{\frac{1}{3}}$.

114. $(a+x)^{\frac{1}{n}} = (x^2+5ax+b^2)^{\frac{1}{2n}}$.

115. $\sqrt[3]{(1+x)} + \sqrt[3]{(1-x)} = \sqrt[3]{2}$.

116. $\dfrac{\sqrt[3]{(x+1)}-\sqrt[3]{(x-1)}}{\sqrt[3]{(x+1)}+\sqrt[3]{(x-1)}} = \dfrac{1}{2}$.

117. $\dfrac{a+x+\sqrt{(2ax+x^2)}}{a+x-\sqrt{(2ax+x^2)}} = b^2$.

118. $\sqrt{(1-x)} + \sqrt{(1-x+\sqrt{1+x})} = \sqrt{(1+x)}$.

119. $\sqrt{(1+a)^2+(1-a)x} + \sqrt{(1-a)^2+(1+a)x} = 2a$.

120. $(a^2-x^2)^{\frac{1}{2}} + x(a^2-1)^{\frac{1}{2}} = a^2(1-x^2)^{\frac{1}{2}}$.

121. $\sqrt{\dfrac{b}{a+x}} + \sqrt{\dfrac{c}{a-x}} = \sqrt[4]{\dfrac{4bc}{a^2-x^2}}$

122. $\left\{\frac{x+1}{x-1}\right\}^{\frac{1}{2}} + \left\{\frac{x-1}{x+1}\right\}^{\frac{1}{2}} = a.$

123. $\sqrt{(4a+x)} = 2\sqrt{(b+x)} - \sqrt{x}.$*

XIII. SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

128. When, in the expression for one unknown quantity, another unknown quantity occurs, it is plain (Art. 122) that the value of the former cannot be determined unless that of the latter be known. Thus, the equation $ax + by = c$ being solved with respect to x, gives $x = \frac{c - by}{a}$: where, since a, b, c, denote known quantities, x will depend for its value upon y; so that, whatever value is assigned to y, x will assume a corresponding value.

Again, if the equation be solved with respect to y, it will give $y = \frac{c - ax}{b}$; and y being expressed in terms of x, it will be dependent upon it, and may have an unlimited number of values, corresponding to those assigned to x. It is plain, therefore, that other conditions must be given, in order to

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* Numerical equations of such general forms as Ex. 69, 71, 72, 84, may readily be solved by substituting particular numbers for the letters, in the value of x: thus, if the equation

$$\frac{3x}{4} + \frac{7x}{15} + \frac{11x}{6} - 366 = 0, \text{ be given,}$$

we have by substitution in Ex. 72,

$$x = \frac{(366+0)\times 360}{270+168+660} = \frac{366\times 360}{1098} = \frac{366\times 20\times 18}{61\times 18} = \frac{366\times 20}{61} =$$

$6 \times 20 = 120.$

Also, $\frac{12}{5x} + \frac{9}{4x} + \frac{20}{7x} + \frac{3}{2x} - 87 = 0$ gives, by Ex. 84.

$$x = \frac{1302 + 1221}{280 \times 87} = \frac{2523}{24360} = \frac{29}{280}.$$

enable us to determine the value of either unknown, and consequently that of the other : such conditions are given by other equations, which hold true simultaneously with the given one, and are hence called *simultaneous* equations.

To illustrate this, let us take the equation

$$x + y = 12.$$

Here, the only condition laid down is, that the sum of the two unknowns must be 12; and it is obvious that the equation can be satisfied by a great many numbers;

If $x = 5$, $y = 7$ If $x = 16$, $y = -4$
$x = 2$, $y = 10$ $x = -3$, $y = 15$
$x = 3\frac{1}{2}$, $y = 8\frac{1}{2}$ $x = 8$, $y = 4$
$x = 2\frac{1}{4}$, $y = 9\frac{3}{4}$ $x = 14\frac{3}{7}$, $y = -2\frac{3}{7}$
&c. &c.

Let us now introduce another condition, namely, that the difference of the same unknown quantities must be 4, or that

$$x - y = 4:$$

then x and y become at once determinate, and are respectively 8 and 4. If the equation $x - y = 4$ be considered by itself, there is an infinite variety of numbers which would satisfy it. Thus,

If $x = 20$, $y = 16$ If $x = -3$, $y = -7$
$x = 7$, $y = 3$ $x = -1\frac{1}{2}$, $y = -5\frac{1}{2}$
$x = \frac{1}{2}$, $y = -3\frac{1}{2}$ $x = 8$, $y = 4$
&c. &c.

Among all the above solutions, there is only one, namely $x = 8$, $y = 4$, which satisfies both equations at once.

129. When each equation contains a separate condition, not fulfilled by another equation, nor derivable from it, the equations are called *independent*. Thus the equations

$$x + y = 12, \quad x - y = 4,$$

are independent of one another, as they involve separate conditions: the equation $2x + 2y = 24$ does not involve any new condition, separate from that given by $x + y = 12$, and is not therefore independent of it.

In like manner, in the following equations,

$$\begin{aligned} x + 2y - 3z &= 15; \\ 3x - 4y + z &= 5; \\ 4x - 2y - 2z &= 20; \end{aligned}$$

the first and second involve separate conditions, but the third is the sum of the other two; it contains therefore no new condition: it is not independent, and consequently is of no use in determining the values of the unknown quantities.

From this we may conclude, that in order to determine the values of unknown quantities, *there must be as many conditions as there are unknown quantities, and therefore as many equations as there are unknown quantities.**

130. From the principles which have been already stated (Art. 128), it will appear that in order to determine the value of each of these unknown quantities, we must obtain in some way a single equation in which one only of the quantities shall be present; the unknown will then be

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* If the independent equations be fewer in number than the unknown quantities, there are too few conditions; and if they are more numerous, they may give conditions which are impossible, or incongruous, that is inconsistent with one another; and, when differently combined, may give different values of the unknown; or, though they should not happen to give such conditions, they are not required for the solution, and are therefore useless.

In a series of connected equations, it is not necessary that each equation should contain all the unknown quantities which are to be determined: but merely that each equation of the series should give a separate condition.

expressed in terms of known quantities (Art. 122), and thus its value will be ascertained. For this purpose recourse is had to various artifices, by means of which, from the series of given equations, other equations are deduced, in conformity with the established axioms and processes of Algebra, until a single equation is found containing but one unknown quantity, the rest being made to disappear. Unknowns which are in this way excluded, are said to be *eliminated.* Thus, taking our former example, $x + y = 12$, $x - y = 4$, we see that the application of Axiom 1, Art. 27, will give us $2x = 16$, an equation which does not contain y; so that y is eliminated, and we find that $x = 8$. Again, if we apply the second Axiom, Art. 28, x will be eliminated, and we shall have $2y = 8$; whence $y = 4$; the values of the unknowns being thus 8 and 4, as already found. But these values may be otherwise deduced. Since, by our supposition, x stands for the same quantity in both equations, find two expressions for x, and put these equal to one another. In this way we have $12 - y = 4 + y$; x is eliminated, and we readily find $4 = y$. Now, since $x + y = 12$, and $y = 4$, we have $x + 4 = 12$, and $x = 8$. The value of x is thus found by substitution; and this may plainly be made in either equation.

The analytical artifices, to which reference has been made above, are embraced in four methods of solution, applied to simultaneous equations of the first degree. Two of these have been just illustrated in the simplest cases. We shall state the four methods in order, as applied to equations containing two unknown quantities, and illustrate each by an example. It is usual to denote the equations of a series by numbers, for the sake of convenience of reference in conducting the solution.

First method; *If the coefficients of one of the unknown quantities be not the same in both equations, multiply or divide* *by such numbers as will render those coefficients the same; destroy the terms thus rendered identical by adding or subtracting the equations, and one of the unknown quantities will be eliminated; the resulting equation will give the value of the other unknown; the value of that which has been eliminated will be found by substitution, or by repeating a similar process on the given equations.*

Let the equations be,

$$5x + 7y = 201 \quad . \quad . \quad . \quad . \quad . \quad (1),$$
$$8x - 3y = 137 \quad . \quad . \quad . \quad . \quad . \quad (2).$$

To eliminate y, multiply each equation by the coefficient of y in the other, and there will be obtained

$$15x + 21y = 603,$$
$$56x - 21y = 959;$$

adding $$71x = 1562,$$
whence $$x = 22.$$

But, from equation (2) $$y = \frac{8x - 137}{3};$$

and by substitution, $$y = \frac{176 - 137}{3} = 13.$$

To eliminate x, we should multiply (1) by 8 and (2) by 5, and subtract; this would give $y = 13$, as before.

131. Second method; *Find an expression for either of the unknown quantities from both equations, equate the expressions thus found* (Euc. Ax. 1), *and from the resulting equation find the value of the unknown which it contains; the value of the other unknown will be found by substitution.*

Taking the same two equations, we have

from (1) $$x = \frac{201 - 7y}{5},$$

from (2) $$x = \frac{137 + 3y}{8}.$$

Now x denotes the same number in both, therefore (Euc. Ax. 1),

$$\frac{201 - 7y}{5} = \frac{137 + 3y}{8},$$

an equation from which x is eliminated, and which gives $y = 13$, and hence $x = 22$.

132. Third method; *Find an expression for either of the unknown quantities in one of the equations, and substitute this expression for that unknown in the other equation; from the resulting equation find the value of the unknown quantity which it contains, and the value of the other will be determined by substitution.*

Taking again the same equations, we have from (1)

$$x = \frac{201 - 7y}{5},$$

and substituting this expression in place of x in (2), we have

$$\frac{1608 - 56y}{5} - 3y = 137;$$

whence $y = 13$, and therefore $x = 22$, as before.

Given $$x + y = a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$
$$x^2 - y^2 = b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Here, from (1) $$x = a - y,$$

which being put in (2) gives $(a - y)^2 - y^2 = b$,

or $$a^2 - 2ay + y^2 - y^2 = b;$$

whence $$y = \frac{a^2 - b}{2a};$$

then, by substitution, $$x = a - \frac{a^2 - b}{2a} = \frac{a^2 + b}{2a}.$$

133. Fourth method; *Multiply one of the equations by any assumed quantity; from the product subtract the other equation, and, in the result, assign such values to the assumed quantity as will make first one, and then the other, unknown quantity vanish.*

Given $2x + 3y = 23 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$

$5x - 2y = 10 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$

Multiplying the terms of (1) by an assumed quantity, m,
we have $2mx + 3my = 23m$;
from this take $5x - 2y = 10$,
and there remains $(2m - 5)x + (3m + 2)y = 23m - 10$.

Now, m being arbitrary, we may take it such that x shall vanish: we may therefore put $2m - 5 = 0$; whence, $2m = 5$, and $m = 2\frac{1}{2}$: we shall therefore have

$$(5 - 5)x + (3 \times 2\tfrac{1}{2} + 2)y = 23 \times 2\tfrac{1}{2} - 10;$$

whence $9\frac{1}{2}y = 47\frac{1}{2}$, and $y = 5$.

Again, to make y disappear from the same equation, we must have $3m + 2 = 0$; therefore $m = -\frac{2}{3}$: hence,

$$(-\tfrac{4}{3} - 5)x = -\tfrac{46}{3} - 10, \text{ and } x = 4.$$

134. With respect to the comparative advantages of these different methods, it may be observed that the first is in general the most convenient: the second and third almost always give rise to fractions, from which the reduced equation must be afterwards cleared. In the first method, also, the addition and subtraction can be performed, when the coefficients are not large numbers, at the same time with the multiplication or division, which shortens the process considerably. The fourth method will often be found the most expeditious.

135. The same methods are applicable if there be three equations and three unknown quantities. *Eliminate, by the preceding methods, one unknown quantity from two of the equations; and then the same unknown from the third equation, and either of the other two; and from the two resulting equations, find the values of the two unknown quantities which they contain.*

Given
$$5x - 6y + 4z = 15 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$7x + 4y - 3z = 19 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$2x + y + 6z = 46. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Here z will be most easily eliminated: to effect this, multiply (1) by 3, (2) by 4, add the results, and there is obtained
$$43x - 2y = 121. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$
Again, doubling (2), and adding the result to (3), we find
$$16x + 9y = 84. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$
Solving (4) and (5) by any of the preceding Rules we find
$$x = 3,\ y = 4\ ;$$
and hence, by substitution in (3), $z = 6$.

136. The following general method, called the method of *indeterminate multipliers*, may be employed with advantage in some cases. The object is to deduce formulæ to be used in particular cases by substituting numbers for the letters. *Multiply two of the equations by any two assumed quantities; from these products subtract the remaining equation; in the result, assign successively such values to the assumed quantities as will make two of the unknowns vanish at once; in the expressions for the unknown quantities, deduced from the three equations thus found, substitute the values of the assumed quantities, determined from the condition that the coefficients must vanish; and the values of the unknowns will be exhibited in known terms.*

Given
$$ax + by + cz = d \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$a'x + b'y + c'z = d' \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$a''x + b''y + c''z = d'' \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Multiply the first by any assumed quantity, m: the second by any other quantity, n: then adding the results, and subtracting the third equation, we find
$$(am + a'n - a'')x + (bm + b'n - b'')y + (cm + c'n - c'')z = dm + d'n - d''. \quad (4)$$

Since m and n are arbitrary, they may be taken of such value that x and y shall vanish from this equation. Now in order that this may take place, their coefficients must become zero; that is, we must have,

$$am + a'n = a'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$bm + b'n = b'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

On this hypothesis, equation (4) will become

$$(cm + c'n - c'')z = dm + d'n - d'';$$

whence,
$$z = \frac{dm + d'n - d''}{cm + c'n - c''}.$$

From equations (5) and (6) m and n can be readily determined by any of the preceding Rules: these values are

$$m = \frac{a''b' - a'b''}{ab' - a'b}.$$

$$n = \frac{ab'' - a''b}{ab' - a'b}.$$

By substituting these values for m and n in that of z, and then reducing, the latter becomes

$$z = \frac{d(b'a'' - a'b'') + d'(ab'' - ba'') - d''(ab' - ba')}{c(b'a'' - a'b'') + c'(ab'' - ba'') - c''(ab' - ba')}.$$

Again, to make x and z vanish, we must have

$$am + a'n = a'',$$

$$cm + c'n = c''.$$

On this supposition we find, from equation (4),

$$(bm + b'n - b'')y = dm + d'n - d'';$$

whence,
$$y = \frac{dm + d'n - d''}{bm + b'n - b''}:$$

the values of m and n, in this case, are

$$m = \frac{a''c' - a'c''}{ac' - a'c}, \quad n = \frac{ac'' - a''c}{ac' - a'c}.$$

Substituting these as before in the value of y, and reducing the compound fraction, that value becomes

$$y = \frac{d(c'a'' - a'c'') + d'(ac'' - ca'') - d''(ac' - ca')}{b(c'a'' - a'c'') + b'(ac'' - ca'') - b''(ac' - ca')}.$$

Lastly, to make y and z disappear, we require the conditions, $bm + b'n = b''$, $cm + c'n = c''$.

Proceeding as before, we shall find

$$x = \frac{dm + d'n - d''}{am + a'n - a''};$$

and by substitution of the values which m and n have in this case

$$x = \frac{d(c'b'' - b'c'') + d'(bc'' - cb'') - d''(bc' - cb')}{a(c'b'' - b'c'') + a'(bc'' - cb'') - a''(bc' - cb')}.$$

137. These three formulæ may be put under more symmetrical forms, by performing the actual multiplications, arranging the terms with respect to the accents upon a, and so as to make the terms alternately positive and negative, and by changing all the signs in the numerator and denominator of the values of x and z, which will not alter those values: the forms thus deduced are

$$x = \frac{db'c'' - dc'b'' + cd'b'' - bd'c'' + bc'd'' - cb'd''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}$$
$$y = \frac{ad'c'' - ac'd'' + ca'd'' - da'c'' + dc'a'' - cd'a''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}.$$
$$z = \frac{ab'd'' - ad'b'' + da'b'' - ba'd'' + bd'a'' - db'a''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}$$

Given
$$7x + 5y + 2z = 79 \quad \ldots\ldots \quad (1)$$
$$8x + 7y + 9z = 122 \quad \ldots\ldots \quad (2)$$
$$x + 4y + 5z = 55 \quad \ldots\ldots \quad (3)$$
to find x, y, z.

Comparing these with the three general equations at the beginning of the last Art., we perceive that in this case

$$a = 7,\ b = 5,\ c = 2,\ d = 79\ ;$$
$$a' = 8,\ b' = 7,\ c' = 9,\ d' = 122\ ;$$
$$a'' = 1,\ b'' = 4,\ c'' = 5,\ d'' = 55.$$

Substituting these numbers in the general formulæ, and actually multiplying, we have

$$x = \frac{2765 - 2844 + 976 - 3050 + 2475 - 770}{245 - 252 + 64 - 200 + 45 - 14} = \frac{-448}{-112} = 4.$$

The values of the two other unknowns, having the same denominator as that of x, are found more readily: they are

$$y = 9,\ z = 3.$$

When one unknown has been determined in this way, the others may be found by the Rules of Arts. 130-134.

138. Either of the Rules of Arts. 135, 136, may be applied, with slight modifications, to equations containing four or more unknown quantities. Thus, by Art. 136, each of the given equations, except the last, may be multiplied by an assumed quantity, as m, n, p, &c.; and from the sum, the last being subtracted, m, n, p, &c. may be determined so as to make all the unknowns but one vanish. The value of each may be thus found; or, according to Art. 135, the unknowns may be eliminated one by one. This latter method is generally to be preferred to any other, in solving particular examples.

Given
$$7x - 2z + 3u = 17 \quad . \ . \ . \ . \ . \ . \quad (1)$$
$$4y - 2z + t = 11 \quad . \ . \ . \ . \ . \ . \quad (2)$$
$$5y - 3x - 2u = 8 \quad . \ . \ . \ . \ . \ . \quad (3)$$
$$4y - 3u + 2t = 9 \quad . \ . \ . \ . \ . \ . \quad (4)$$
$$3z + 8u = 33 \quad . \ . \ . \ . \ . \ . \quad (5)$$

Here, eliminating z from equations (1) and (5), we find

$$21x + 25u = 117\ ;$$

whence, by division, $x = \dfrac{117 - 25u}{21}$.

Substituting this value of x in equation (3), we obtain

$$5y - \frac{117 - 25u}{7} - 2u = 8\,;$$

which becomes, by reduction,

$$35y + 11u = 173. \quad . \quad . \quad (6)$$

Again, in order to find another equation which may be combined with this one, so as to determine either u or y, substitute in equation (2) the value of z, deduced from equation (5); and, after reduction, there will be obtained

$$12y + 16u + 3t = 99.$$

Combining this with equation (4), and eliminating t, there is obtained $12y + 41u = 171.$ . . (7)

From this last equation and (6), we readily find the value of u to be $u = 3$;
whence, from (6), $y = 4$;
also, from equation (5), $z = \frac{1}{3}(33 - 24) = 3$;
from equation (2) or (4), $t = 1$;
and from (1) or (3), $x = 2$.

EXERCISES.

1. $x + y = 5,$
$x - y = 1.$

2. $x + y = 19,$
$y - x = 5.$

3. $3x - 2y = 14,$
$x + y = 13.$

4. $4x + 2y = 26,$
$2x - y = 3.$

5. $3y + 6x = 15,$
$6x - 3y = 9.$

6. $5y - 5x = 15,$
$3x + 5y = 71.$

7. $-4x + 3y = 13,$
$2y - 6x = -8.$

8. $5x - 2y + 4 = 0,$
$4x + y - 28 = 0.$

9. $10x = 2 + 2y,$
$4y = 20 - 4x.$

10. $ax + by = c,$
$dx + fy = h.$

11. $x + y = s,$
$x - y = d.$

12. $3x + 2y = 4,$
$4y - 3x + 1 = 0.$

13. $ax = by,$
$x + y = c.$

14. $5v - 2z = \frac{1}{3},$
$6z - 6v = 5.$

15. $x + ny = p,$
$y + nx = q.$

16. $\frac{x}{2} + \frac{y}{4} = 7.$
$\frac{x}{3} + \frac{y}{3} = 6.$

17. $3x - 7 = 4 + x + y,$
$2y + 79 = 5x.$

18. $px + y = 1,$
$x - py = 2.$

19. $2x + \cdot 4y = 1 \cdot 2,$
$3 \cdot 4x - \cdot 02y = \cdot 01.$

20. $ax + by = n,$
$4by - 2ax = m.$

21. $x + 1 = \sqrt{(x^2 + 2y - 1)}$
$y + 1 = \sqrt{(3x + y^2)}$

22. $\frac{100 - x}{3} = \frac{100 + y}{4}$
$\frac{50 - x}{4} = \frac{50 + y}{7}.$

23. $\frac{x}{8} + 8y = 194,$
$\frac{y}{8} + 8x = 131.$

24. $\frac{m}{x} + \frac{n}{y} = a,$
$\frac{n}{x} + \frac{m}{y} = b.$

25. $\frac{a}{x} + \frac{b}{y} = m,$
$\frac{c}{x} + \frac{d}{y} = n.$

26. $x + y + \frac{3}{10} = 1,$
$2 - 5y + 10x = 0.$

27. $2x + 3y = \frac{4a}{b},$
$y - x = \frac{a}{12b}.$

28. $\frac{1}{x} + \frac{1}{y} - 7 = 3,$
$\frac{2}{x} - \frac{2}{y} + 7 = -5.$

29. $x + \sqrt{(x^2 + y)} = a,$
$x + \sqrt{(x^2 - y)} = b.$

30. $\frac{3x - 1}{5} + 3y - 4 = 15,$
$\frac{3y - 5}{6} + 2x - 8 = 7\frac{2}{3}.$

31. $\frac{a}{b + y} = \frac{b}{3a + x},$
$ax + 2by = c.$

32. $x + y : a :: x - y : b,$
$x^2 - y^2 = c.$

33. $\frac{x}{a} + \frac{y}{b} = 1 - \frac{x}{c},$
$\frac{y}{a} + \frac{x}{b} = 1 + \frac{y}{c},$

34. $\frac{11x - 5y}{22} = \frac{3x + y}{32}$
$8x - 5y = 1$

35. $(x+5)(y+7)=(x+1)(y-9)+112,$

$2x+10=3y+1.$

36. $\dfrac{4x-8y+5}{2}=\dfrac{10x^2-12y^2-14xy+2x}{5x+3y+3}+2,$

$\sqrt{(6+x)}:\sqrt{(6-y)}::3:2.$

37. $5x+3y=\dfrac{(8a-2b)ab}{a^2-b^2},$

$a^2y-\dfrac{acb^2}{a+b}+(a+b+c)bx=b^2y+(a+2b)ab.$

38. $\dfrac{3x-5y}{3}-\dfrac{2x-8y-9}{12}=\tfrac{1}{2}y+\tfrac{1}{3}+\tfrac{1}{4},$

$\dfrac{x}{7}+\dfrac{y}{4}+1\tfrac{1}{3}:4x-\dfrac{y}{8}-24::3\tfrac{1}{3}:3\tfrac{1}{2}.$

39. $x+y=18{\cdot}73,$

$0{\cdot}56x+13{\cdot}8121y=763{\cdot}4.$

40. $bcx=cy-2b,$

$b^2y+\dfrac{a(c^3-b^3)}{bc}=\dfrac{2b^3}{c}+c^3x.$

41. $4x-34\tfrac{1}{3}-\dfrac{4y+13x}{27-6y}=\dfrac{12x+8}{3},$

$3x+\dfrac{21-4y}{4x-10}=\dfrac{18x+13}{6}-2\tfrac{1}{9}.$

42. $\dfrac{7x+6}{11}+\dfrac{4y-9}{3}=3x-\dfrac{13-x}{2}-\dfrac{3y-x}{5},$

$3x+4:2y-3::5:3.$

43. $\dfrac{\dfrac{7x}{4}+6y}{5}-\dfrac{\dfrac{3y+6}{5}-\dfrac{3x-2}{10}}{8}=5-\dfrac{x}{16}$

$\dfrac{3x}{2}+\dfrac{2y}{3}+2\tfrac{1}{2}:\dfrac{x}{2}-\dfrac{y}{3}+\dfrac{1}{6}::10\tfrac{1}{2}:1\tfrac{1}{6}$

44. $$4+\frac{12y-\frac{6y+2}{x}}{11}=y+\frac{\frac{3xy-31}{11}+10x+13}{3x},$$

$$\frac{2x}{3}-\frac{3x-5}{y+7}=\frac{4xy+\frac{170}{8}}{6y+27}.$$

45. $$\frac{4x-2y+3}{3}-\frac{18-x+5y}{7}=\frac{x}{4}-\frac{y}{5}-\frac{1}{7}-7\tfrac{7}{10}.$$

$$2x-y+15 : y-2x+15 : : \frac{x}{3}-\frac{y}{4}+\tfrac{3}{4} : \frac{y}{4}-\frac{x}{3}+\frac{1}{12}.$$

46. $$\sqrt{y}-\sqrt{(20-x)}=\sqrt{(y-x)}$$

$$\sqrt{(y-x)}+\sqrt{(20-x)} : \sqrt{(y-x)}=5 : 3.$$

47. $$\begin{aligned}x+y-z&=23,\\ x-y+z&=17,\\ x+y+z&=47.\end{aligned}$$

48. $$\begin{aligned}x+y+z&=33,\\ y-x+z&=23,\\ z-x-y&=1.\end{aligned}$$

49. $$\begin{aligned}x+y+z&=31,\\ x+y-z&=25,\\ x-y-z&=9.\end{aligned}$$

50. $$\begin{aligned}x+y+z&=29\tfrac{1}{4},\\ x+y-z&=18\tfrac{1}{4},\\ x-y+z&=13\tfrac{3}{4}.\end{aligned}$$

51. $$\begin{aligned}x+y+z&=53,\\ x+2y+3z&=107,\\ x+3y+4z&=137.\end{aligned}$$

52. $$\begin{aligned}x+2y+z&=16,\\ \frac{x}{3}+\frac{y}{2}+\frac{z}{5}&=4,\\ 2x+4y-3z&=7.\end{aligned}$$

53. $$\begin{aligned}v+4u+3w&=18,\\ 3v-2u+4w&=11,\\ 4v+6u-5w&=1.\end{aligned}$$

54. $$\begin{aligned}2v+4u-3w&=22,\\ 4v-2u+5w&=18,\\ 6v+7u-w&=63.\end{aligned}$$

55. $$\begin{aligned}x+y&=10,\\ x+z&=19,\\ y+z&=23.\end{aligned}$$

56. $$\begin{aligned}x-z&=1,\\ 2y+z&=11,\\ 2x+3y&=23.\end{aligned}$$

57. $$\begin{aligned}x+y+z&=30,\\ 8x+4y+2z&=50,\\ 27x+9y+3z&=64.\end{aligned}$$

58. $$\begin{aligned}4z+12y+7x&=128,\\ 7z+3y+3x&=60,\\ 5z+y+6x&=68.\end{aligned}$$

59. $$\begin{aligned}4z-5y&=35,\\ 7x-2y&=22,\\ 2z+x&=68.\end{aligned}$$

60. $$\begin{aligned}3x-41-9y+8z&=0,\\ -5x+4y+20+2z&=0,\\ 11x-7y-37-6z&=0.\end{aligned}$$

61. $3z = 2x$,
$2x + y = 34$,
$y + z = 18$.

62. $\dfrac{x+y}{3} + 2z = 21$,
$\dfrac{y+z}{2} - 3x = -65$,
$\dfrac{3x+y-z}{2} = 38$.

63. $\dfrac{1}{x} + \dfrac{1}{y} = a$
$\dfrac{1}{x} + \dfrac{1}{z} = b$,
$\dfrac{1}{y} + \dfrac{1}{z} = c$.

64. $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{3}{4}$,
$\dfrac{1}{x} + \dfrac{1}{z} = \dfrac{2}{3}$,
$\dfrac{1}{y} + \dfrac{1}{z} = \dfrac{5}{12}$.[*]

65. $x + \frac{1}{2}y + \frac{1}{3}z = 32$,
$\frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z = 15$,
$\frac{1}{4}x + \frac{1}{5}y + \frac{1}{6}z = 12$.

66. $my = nx$,
$pz = qx$,
$x + y + z = a$.

67. $ax + by = c$,
$dx + ey = f$,
$gy + hz = l$.

68. $x + y = a$,
$x + z = b$,
$y + z = c$.[†]

69. $x - 9y + 3z - 10u = 21$,
$2x + 7y - z - u = 683$,
$3x + y + 5z + 2u = 195$,
$4x - 6y - 2z - 9u = 516$.

70. $x - y = 2$,
$x - z = 3$,
$y + z = 9$.

71. $\dfrac{xy}{x+y} = 1$,
$\dfrac{xz}{x+z} = 2$.
$\dfrac{yz}{y+z} = 3$.

72. $5x - 11\sqrt{y} + 13\sqrt[3]{z} = 22$,
$4x + 6\sqrt{y} + 5\sqrt[3]{z} = 31$,
$x - \sqrt{y} + \sqrt[3]{z} = 2$,

73. $9x - 2z + u = 41$,
$7y - 5z - t = 12$,
$4y - 3x + 2u = 5$,
$3y - 4u + 3t = 7$,
$7z - 5u = 11$.

74. $xy = a(x + y)$,
$xz = b(x + z)$,
$yz = c(y + z)$.

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[*] This equation may also be solved by substituting for a, b, &c. in the last equation, their numerical values in this one.

[†] This equation is most easily solved, by subtracting each equation severally from the half-sum of the three.

75. $x(x+y+z)=a^2$,
$y(x+y+z)=b^2$,
$z(x+y+z)=c^2$.

76. $xyz = 231$,
$xyw = 420$,
$yzw = 1540$,
$xzw = 660$.

77. Shew that the following equations are not sufficient to determine the unknown quantities contained in them, either from being *incongruous*, or *not independent*.

1. $3x-2y=14$,
$6x-4y=28$.

2. $x+y=7$,
$3x+3y=30$.

3. $6x+9y=27$,
$8x+12y=36$.

4. $x+y=7$,
$3x-y=1$,
$x+2y=10$.

5. $x+3y+4z=9$,
$3x-2y+17z=25$,
$x+14y-z=11$.

6. $2x+3y-z=16$.
$5x-2y+7z=21$.
$3x-5y+8z=12$.

7. $3x-2y+5z=14$,
$2x+y-8z=10$,
$8x-3y+2z=38$.

XIV.—QUADRATIC EQUATIONS,

CONTAINING ONE UNKNOWN QUANTITY.

139. A *quadratic* equation, or an equation of the *second degree*, is one in which the unknown quantity is in the second power.

140. A quadratic which contains the second power only of the unknown quantity, is called a *pure* quadratic; while one that contains both its second and first powers, is called a *compound* or *adfected* quadratic. Thus, a pure quadratic is of the form $ax^2 = \pm b$; while a compound quadratic is of the form $ax^2 \pm bx = \pm c$.

141. It is usual to regard as a compound quadratic, any equation which contains two such powers of the unknown

quantity, that the index of one of them is double that of the other; that is to say, any equation of the form $ax^{2n} \pm bx^{n} = \pm c$.*

142. That quantity which is independent of the unknown, is called the *absolute* term of the equation; thus, in the above equations, c is the absolute term.

143. To solve a pure quadratic; *Simplify the equation by applying Art.* 127, *and extract the square root of both members.*

For, the general form being $ax^2 = \pm c$, where c represents the aggregate of all the known terms, we have by division,

$$x^2 = \pm \frac{c}{a} \text{ and } x = \pm \sqrt{\pm \frac{c}{a}}$$

Given $7x^2 + 11 = 56 + 2x^2$ to find x.

By transposition $5x^2 = 45$,

whence, $x^2 = 9$,

and $x = \pm 3$.

It is unnecessary to use the double sign on the left member; for if we put $\pm x = \pm 3$, we shall have no more than two values, since $-x = +3$ is the same as $+x = -3$, and $-x = -3$ is $x = 3$; so that the double sign need only be used on one side.

144. To solve a compound quadratic. First method; *Simplify the equation, by applying Art.* 127; *divide both*

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[*] A simple equation is often defined to be one which contains only one power of the unknown quantity; and hence those equations which contain more than one power are called compound. In this view, all pure equations of the second, third, &c. degrees, are considered simple equations, and solved as such. We have, however, considered them as belonging more properly to another class of equations.

members by the coefficient of the higher power of the unknown quantity; add to both sides the square of half the coefficient of the lower power of the unknown quantity; extract the square root of both sides; and, from the resulting simple equation, determine the values of the unknown quantity.

The reason of this rule may be explained as follows: dividing both members of the general equation

$$ax^2 \pm bx = \pm c,^*$$

by a, and putting for brevity, $\frac{b}{a} = p$, $\frac{c}{a} = q$,

we obtain $$x^2 \pm px = \pm q.$$

Now, the terms of the left member are plainly the two first terms of the square of a binomial, whose first term is x (Art. 97); and since px is *twice* the product of the two terms of the binomial, the second term must be the quantity of which p is double, that is, it must be $\frac{1}{2}p$: the square of this is $\frac{1}{4}p^2$, which, being added to both sides, gives

$$x^2 \pm px + \tfrac{1}{4}p^2 = \pm q + \tfrac{1}{4}p^2.$$

The left member is now a complete square; therefore, by evolution and Art. 96, we have

$$x \pm \tfrac{1}{2}p = \pm \sqrt{(\pm q + \tfrac{1}{4}p^2)};$$

and by transposition,

$$x = \mp \tfrac{1}{2}p \pm \sqrt{(\pm q + \tfrac{1}{4}p^2)}.^\dagger$$

---

* The equation $ax^2 \pm bx = \pm c$, is no less general than $\pm ax^2 \pm bx = \pm c$; because, if the first term be negative, it can be rendered positive by changing the signs of all the terms.

† This formula, expressed in words, gives the following rule:

Having prepared the equation by reduction, as in the rule in the text: to the half coefficient of the lower power of the unknown, with its sign changed, annex by the signs ± the square root of the sum of the absolute term, and the square of the half coefficient.

Thus, to find x from the equation $x^2 - 10x = 24$, we have $p = 10$ and $q = 24$, and therefore $x = 5 \pm \sqrt{(24 + 25)} = 12$, or -2.

Since $\frac{1}{2}p$, and the quantity under the radical sign, are both affected by the double sign, it is plain that the above general formula includes the following cases :

1. When $x^2 + px = + q,\quad x = \begin{cases} -\frac{1}{2}p + \sqrt{(q + \frac{1}{4}p^2)}, \\ \quad\text{or} \\ -\frac{1}{2}p - \sqrt{(q + \frac{1}{4}p^2)}. \end{cases}$

2. When $x^2 - px = + q,\quad x = \begin{cases} \frac{1}{2}p + \sqrt{(q + \frac{1}{4}p^2)}, \\ \quad\text{or} \\ \frac{1}{2}p - \sqrt{(q + \frac{1}{4}p^2)}. \end{cases}$

3. When $x^2 + px = - q,\quad x = \begin{cases} -\frac{1}{2}p + \sqrt{(- q + \frac{1}{4}p^2)}, \\ \quad\text{or} \\ -\frac{1}{2}p - \sqrt{(- q + \frac{1}{4}p^2)}. \end{cases}$

4. When $x^2 - px = - q,\quad x = \begin{cases} \frac{1}{2}p + \sqrt{(- q + \frac{1}{4}p^2)}, \\ \quad\text{or} \\ \frac{1}{2}p - \sqrt{(- q + \frac{1}{4}p^2)}. \end{cases}$

In the third and fourth of these general values, if $\frac{1}{4}p^2 < q$, the quantity within the vinculum will be negative, and therefore (Art. 96) the values of x will be impossible, or imaginary. The first and second forms will always give real values.

Given $\frac{5x^2}{12} + \frac{2}{3}x + 6\frac{1}{4} = \frac{3}{4}x + 20 - \frac{10x}{3}$, to find the value of x.

Here by reduction, $\quad x^2 + 8x = 33$;

then the square will be completed by adding to both sides the square of the half coefficient of x, that is 16 ; we thus get $\quad x^2 + 8x + 16 = 33 + 16,$

or, $\quad x^2 + 8x + 16 = 49$;

hence $\quad x + 4 = \pm 7$

and $\quad x = \pm 7 - 4$;

therefore, $\quad x = 3, \text{ or } - 11.$

Applying the rule in the note, we have,

$$x = -4 \pm \sqrt{(33+16)} = -4 \pm \sqrt{49} = -4 \pm 7 = 3, \text{ or } -11.$$

Verifying this equation, we find, $9 + 24 = 33$,

and $121 - 88 = 33$.

145. Second Method; *Simplify the equation, by applying Art. 127; multiply both members by four times the coefficient of the higher power of the unknown quantity; add to both members the square of the coefficient of the lower power of the unknown quantity in the original equation; extract the square root of both sides; and, from the resulting simple equation, find the values of the unknown quantity.*

Resuming the general equation $ax^2 \pm bx = \pm c$,
and multiplying both members by $4a$, we obtain

$$4a^2x^2 \pm 4abx = \pm 4ac.$$

Now, $4abx$ must be the product of the first term of the root, $2ax$, by twice the second; hence the second must be b; and if b^2 be added to both sides, the left will be a complete square; we thus get $4a^2x^2 \pm 4abx + b^2 = \pm 4ac + b^2$;
by evolution, $2ax \pm b = \pm \sqrt{(\pm 4ac + b^2)}$;
transposing and dividing, we find

$$x = \frac{\mp b \pm \sqrt{(\pm 4ac + b^2)}}{2a}.$$ [*]

It is easy to see that if in the general equation the second term be even, that is, if $ax^2 + 2bx = c$ be the form, it will be sufficient to multiply by a, since we shall then have $a^2x^2 + 2abx + b^2 = ac + b^2$, where the left member is a complete square.

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[*] This method of solving quadratics is taken from the "Bija Ganita," a Hindoo Treatise on Algebra, of the date of about the beginning of the thirteenth century, and made known in Europe, through a translation from a Persian copy, by Mr. Edward Strachey, of the Bengal Civil

Given $x^2 + \frac{2}{3}x - 3 = 25\frac{1}{3}$, to find the values of x.

Simplifying, we have

$$3x^2 + 2x = 85\,;$$

then multiplying by 4 × 3, or 12, we find

$$36x^2 + 24x = 1020,$$

and $$36x^2 + 24x + 4 = 1024\,;$$

hence by evolution $$6x + 2 = \pm 32\,;$$

therefore, $$6x = 30, \text{ or } -34,$$

and $$x = 5, \text{ or } -5\tfrac{2}{3}.$$

Service. An account of this curious work, and of another treatise on collateral subjects, called "The Leelawuttee," will be found in Dr. Hutton's Mathematical Tracts, vol. ii., pp. 151-179.

Subsequently, the late Dr. Thomson, Professor of Mathematics in the University of Glasgow, not having seen or heard of the Hindoo mode of solution, published, in a Belfast periodical, the following rule, which, though the demonstration of it is the same as that in the text, is yet, in a practical point of view, quite different from the Hindoo rule, as it gives the values of x without any intermediate work, such as completing the square, extracting the root, &c. It was not till several years after that Dr. Thomson became acquainted with the Hindoo method.

"Reduce the given equation to the form $ax^2 + bx = c$; then, to find x, to the coefficient of the second term with its sign changed, annex by the double sign ±, the square root of the quantity obtained, by adding together the square of that coefficient and four times the product of the absolute term and the coefficient of the first term; and divide the whole expression thus obtained by twice this last-mentioned coefficient."

Applying this rule to the equation $3x^2 + 5x = 42$, we have at once

$$x = \frac{-5 \pm \sqrt{(25 + 504)}}{6} = \frac{-5 \pm \sqrt{529}}{6} = \frac{-5 \pm 23}{6} = 3, \text{ or } -4\tfrac{2}{3}.$$

The second rule may also be immediately derived from the first. Putting the original equation under the form

$$x^2 \pm \frac{b}{a}x = \pm\frac{c}{a},$$

and completing the square, by adding to both sides the square of the half coefficient of x, that is $\frac{b}{2a}$, we obtain

$$x^2 \pm \frac{b}{a}x + \frac{b^2}{4a^2} = \pm\frac{c}{a} + \frac{b^2}{4a^2};$$

Applying the Rule in the Note, we should have

$$x = \frac{-2 \pm \sqrt{(1020+4)}}{6} = \frac{-2 \pm 32}{6} = 5, \text{ or } -5\tfrac{2}{3}.$$

146. By employing the first of the two preceding Rules, fractions will be introduced into the operation, whenever, in equations reduced to the form $x^2 \pm px = \pm q$, the coefficient p is odd or fractional. In such cases the second Rule should be used, as fractions will not occur till the last. The first Rule is preferable when, in equations of the form $ax^2 \pm bx = \pm c$, a is a large number, as the operation by the second Rule is then tedious and troublesome.

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hence, by evolution,

$$x \pm \frac{b}{2a} = \pm \sqrt{(\pm \frac{c}{a} + \frac{b^2}{4a^2})}.$$

Adding the fractions under the radical sign, extracting the root of the denominator, and then transposing, we find

$$x = \frac{\mp b \pm \sqrt{(\pm 4ac + b^2)}}{2a}$$

The following method of deriving the same formula, will be a useful exercise in managing surds. In the original equation $ax^2 \pm bx = \pm c$, it is plain that $ax^2 = (x\sqrt{a})^2$; so that the first term of the binomial is $x\sqrt{a}$: also that $bx = (x\sqrt{a} \times \frac{b}{2\sqrt{a}}) \times 2$; so that the second term of the binomial is $\frac{b}{2\sqrt{a}}$: hence, $\frac{b^2}{4a}$ will complete the square; we therefore have

$$(x\sqrt{a})^2 \pm 2x\sqrt{a}\frac{b}{2\sqrt{a}} + \frac{b^2}{4a} = \pm c + \frac{b^2}{4a};$$

hence, by evolution,

$$x\sqrt{a} \pm \frac{b}{2\sqrt{a}} = \pm \sqrt{(\pm c + \frac{b^2}{4a})}.$$

Transposing, and dividing by $\sqrt{a}$, we find

$$x = \mp \frac{b}{2a} \pm \sqrt{(\pm \frac{c}{a} + \frac{b^2}{4a^2})};$$

and by reduction,

$$x = \frac{\mp b \pm \sqrt{(\pm 4ac + b^2)}}{2a}$$

the same as by the two former methods.

EXAMPLES.

1. Given $6x - 30 = 3x^2$, to find the values of x.

By transposition and division,

$$x^2 - 2x = -10\,;$$

therefore by Rule 1.

$$x^2 - 2x + 1 = -9.$$

By evolution, $\quad x - 1 = \pm\sqrt{(-9)}\,;$

whence $\quad x = 1 \pm 3\sqrt{-1}\,;$

both of which values are imaginary, as is obvious from the general values 3 and 4, Art. 144.

2. Given $5x^2 + 4x = 25$; required x.

Here Rule II. is to be preferred ; therefore,

$$100x^2 + 80x + 16 = 516.$$

Evolving, transposing, and dividing, we have

$$x = \frac{-4 \pm \sqrt{516}}{10} = 1{\cdot}87516\ldots \text{ or } -2{\cdot}67156\ldots.$$

3. Given $\sqrt{(x+21)} + \sqrt[4]{(x+21)} = 12$, to find the values of x.

Completing the square, we have

$$\sqrt{(x+21)} + \sqrt[4]{(x+21)} + \tfrac{1}{4} = 12\tfrac{1}{4} = \tfrac{49}{4}\,;$$

by evolution, $\quad \sqrt[4]{(x+21)} + \tfrac{1}{2} = \pm\tfrac{7}{2}.$

whence $\quad \sqrt[4]{(x+21)} = 3, \text{ or } -4\,;$

whence $\quad x = 60, \text{ or } 235.$

4. Given $3x - \dfrac{3x-10}{9-2x} = 2 + \dfrac{6x^2-40}{2x-1}$, to determine x.

Multiplying by $2x - 1$, and simplifying, and again by $9 - 2x$, and simplifying, we find

$$8x^2 - 124x = -368\,;$$

by division, $x^2 - \frac{31}{2}x = -46.$

Applying Rule I, which is in this case the simpler, we find

$$x^2 - \frac{31}{2}x + \frac{961}{16} = -46 + \frac{961}{16} = \frac{225}{16};$$

by evolution $x - \frac{31}{4} = \pm\frac{15}{4};$

whence $x = 4$, or $11\frac{1}{2}$.

5. Given $\frac{x + \sqrt{(x^2 - 9)}}{x - \sqrt{(x^2 - 9)}} = (x - 2)^2$, to find the values of x.

Here (Art. 117),

$$\frac{(x + \sqrt{x^2 - 9})^2}{9} = (x - 2)^2;$$

by evolution,

$$\frac{x + \sqrt{(x^2 - 9)}}{3} = \pm(x - 2).$$

Taking the right member positive, and making various reductions, we find $x^2 - 8x = -15;$
which gives $x = 5$, or 3.

Again, taking the right member negative, and reducing as before, there results

$$x = \frac{8 \pm \sqrt{11}\sqrt{-1}}{5}:$$

these two values are imaginary; the two others are real.

6. Given $2x^2 + 3x - 5\sqrt{(2x^2 + 3x + 9)} + 3 = 0$, to find x.

Here the quantity under the radical sign exceeds the quantity outside of it, by 6; therefore, if 6 be added to both members, the first unknown term will be the square of the second: adding, we have

$$2x^2 + 3x + 9 - 5\sqrt{(2x^2 + 3x + 9)} = 6,$$

solving this equation by the usual method, and then squaring, we have $2x^2 + 3x + 9 = 36$, or 1.

The first of these gives $x = 3$, or $-4\frac{1}{2}$;

and the second, $x = \dfrac{-3 \pm \sqrt{(-55)}}{4}$.

7. Given $x^4 - 2x^3 + x = 132$.

It is plain that the equation will not be changed, by adding x^2 to the left member, and then subtracting it; this gives

$$x^4 - 2x^3 + x^2 - x^2 + x = 132,$$

or

$$(x^2 - x)^2 - (x^2 - x) = 132.$$

Solving this equation, we find, for the two values,

$$x^2 - x = 12, \text{ and } x^2 - x = -11:$$

the first equation gives

$$x = 4, \text{ or } -3;$$

the second,

$$x = \frac{1 \pm \sqrt{43}\sqrt{(-1)}}{2}.$$

8. Given, $\left(x^2 - \dfrac{a^4}{x^2}\right)^{\frac{1}{2}} + \left(a^2 - \dfrac{a^4}{x^2}\right)^{\frac{1}{2}} = \dfrac{x^2}{a}$, to find x.

By transposition,

$$\left(x^2 - \frac{a^4}{x^2}\right)^{\frac{1}{2}} - \frac{x^2}{a} = -\left(a^2 - \frac{a^4}{x^2}\right)^{\frac{1}{2}};$$

squaring, transposing, and reducing by Art. 75, we find

$$x^2 - \frac{2x^2}{a}\sqrt{\left(\frac{x^4 - a^4}{x^2}\right)} + \frac{x^4 - a^4}{a^2} = 0,$$

that is,

$$x^2 - \frac{2x}{a}\sqrt{(x^4 - a^4)} + \frac{x^4 - a^4}{a^2} = 0,$$

which is a complete square. Evolving, transposing, clearing of fractions, and squaring, we find

$$a^2x^2 = x^4 - a^4;$$

whence,

$$x^4 - a^2x^2 = a^4.$$

Solving this quadratic, we find

$$2x^2 = a^2 \pm a^2\sqrt{5} = a^2(1 \pm \sqrt{5}),$$

and $$x = \pm a\sqrt{\left(\frac{1 \pm \sqrt{5}}{2}\right)}.$$

9. Given $\frac{a-x}{x} + \frac{x}{a-x} = \frac{b}{c}$; required the values of x.

Multiplying all the terms by $\frac{a-x}{x}$, and transposing, we find

$$\left(\frac{a-x}{x}\right)^2 - \frac{b}{c}\cdot\frac{a-x}{x} = -1.$$

Completing the square, and extracting the root, we have

$$\frac{a}{x} - 1 = \frac{b}{2c} \pm \frac{\sqrt{(b^2 + 4c^2)}}{2c};$$

therefore, $$x = \frac{2ac}{2c + b \pm \sqrt{(b^2 + 4c^2)}}.$$

10. Given $\frac{49x^2}{4} + \frac{48}{x^2} - 49 = 9 + \frac{6}{x}$; required x.

Here, it is easy to see that the square will be completed on both sides, by adding $\frac{1}{x^2}$: this gives

$$\frac{49x^2}{4} - 49 + \frac{49}{x^2} = 9 + \frac{6}{x} + \frac{1}{x^2};$$

evolving and reducing, we find

$$7x^2 - 6x = 16, \text{ and } 7x^2 + 6x = 12:$$

the solution of these two equations gives

$$x = 2, \text{ or } -\tfrac{8}{7};$$

and

$$x = \frac{-3 \pm \sqrt{93}}{7} = 0{\cdot}949 \ldots \text{ or } -1{\cdot}8062 \ldots.$$

11. Let $x^3 + 1 = 0$, required x. Resolving this into its factors, we have $(x + 1)(x^2 - x + 1) = 0$. Now this condition is satisfied, either when $x + 1 = 0$, or when $x^2 - x + 1 = 0$. The former gives $x = -1$, and the latter $x = \frac{1 \pm \sqrt{-3}}{2}$; that is, there are three values of the unknown.

EXERCISES.

1. $\frac{3x^2}{5} - 1 = 8\frac{3}{5}$
2. $\frac{5x^2}{2} + x^2 = 126$
3. $x^2 - 17 = 130 - 2x^2$
4. $3x^2 - 4 = 28 + x^2$
5. $y^2 = 9a^6$
6. $x^2 + ab = 5x^2$
7. $x^2 + 6x = 40$
8. $x^2 + 12x = 108$
9. $x^2 + 14x = 51$
10. $x^2 + 10x = 24$
11. $x^2 + 6x = 55$
12. $x^2 - 6x = 55$
13. $x^2 + 8x - 33 = 0$
14. $x^2 - 8x - 33 = 0$
15. $x^2 - 20x - 69 = 0$
16. $x^2 - 20x + 84 = 0$
17. $x^2 + 12x + 35 = 0$
18. $x^2 + 16x + 48 = 0$
19. $5x^2 - 10x = 75$
20. $3x^2 - 48x = -180$
21. $2x^2 + 26x + 40 = -20$
22. $\frac{4x^2}{5} + 8x - 50\frac{1}{4} = 429\frac{3}{4}$
23. $3x^2 - 72x = -132$
24. $2x^2 + 32x + 110 = 0$
25. $2x^2 - 3x = 54$
26. $3x^2 - 7x = 20$
27. $3x^2 - 5x = 28$
28. $5x^2 - 4x = 156$
29. $3x^2 - 5x = 2$
30. $6x^2 + 5x = 4$
31. $2x^2 + 3x - 2 = 0$
32. $6x^2 - 5x + 1 = 0$
33. $x^2 - 13x + 40 = 0$
34. $x^2 - x = 42$
35. $6x^2 + x = 2$
36. $4x^2 - 20x = 336$
37. $5x^2 + 35x = 600$
38. $x = 1 + \frac{110}{x}$
39. $x^2 - \frac{3}{10}x = \frac{1}{10}$
40. $17x^2 + 19x = 1848$
41. $\frac{x^2}{10} + 350 - 12x = 0$
42. $x^2 - x = 11342$
43. $x - \frac{x^3 - 8}{x^2 + 5} = 2$
44. $x^2 - 8x = 14$

45. $2x^2 + 15 = 3x$

46. $x^2 - 6x + 19 = 13$

47. $4x - \frac{36 - x}{x} - 46 = 0$

48. $\frac{4x}{9} + \frac{x - 5}{x + 3} = \frac{4x + 7}{19}$

49. $\frac{16}{x} - \frac{100 - 9x}{4x^2} - 3 = 0$

50. $\frac{4}{x - 3} - \frac{3}{x + 5} = \frac{1}{18}$

51. $\frac{2x - 3}{3x - 5} + \frac{3x - 5}{2x - 3} = \frac{5}{2}$

52. $x + 2\sqrt{x} = 24$

53. $2x^4 - 7x^2 - 99 = 0$

54. $\sqrt{x^5} + \sqrt{x^3} = 6\sqrt{x}$

55. $\frac{\sqrt{4x} + 2}{4 + \sqrt{x}} = \frac{4 - \sqrt{x}}{\sqrt{x}}$

56. $\sqrt{x^3} - 2\sqrt{x} = x$

57. $\frac{8}{x^3} + 2 - \frac{17}{x^{\frac{3}{2}}} = 0$

58. $\sqrt[5]{x} + \sqrt[5]{x^2} = 6$

59. $(x - 5)^3 - 3(x - 5)^{\frac{3}{2}} = 40$

60. $x^6 + 3x^3 = 810$

61. $5x^6 - 90x^3 - 270 = 945$

62. $x^4 - 74x^2 + 1225 = 0$

63. $x^4 + x^3 - 4x^2 + x + 1 = 0$

64. $\frac{x}{a} + \frac{a}{x} - \frac{2}{a} = 0$

65. $\frac{2x(a - x)}{3a - 2x} = \frac{a}{4}$

66. $x^2 - \frac{2}{3x} = 1\frac{4}{9}$

67. $x^6 - 7x^3 = 8$

68. $\frac{\frac{3\sqrt{x}}{5} - 2}{x - 5} = \frac{1}{20}$

69. $\sqrt{(x+5)} = \frac{12}{\sqrt{(x + 12)}}$

70. $3x^2 - x^{\frac{7}{6}} = 3104x^{\frac{1}{3}}$

71. $x^3 - 3x = 2$

72. $x^n + x^{-n} = a$

73. $(a + b)x^2 - cx = \frac{ac}{a + b}$

74. $x^3 - 1 = 0$

75. $ax^{-\frac{5}{4}} - bx^{-\frac{5}{8}} = c$

76. $x^4 + 1 = 0$

77. $x^4 - 3x^3 + 3x + 1 = \frac{4x^2}{9}$

78. $2x\sqrt{(1 - x^4)} = a(1 + x^4)$

79. $\frac{8x - 20}{2x + 10} - \frac{7 - 3x}{x} = \frac{14}{4}$

80. $\frac{1}{x^2 - 3x} + \frac{1}{x^2 + 4x} = \frac{9}{8x}$

81. $\frac{2x}{15} + \frac{3x + 50}{3(10 - x)} = \frac{12x - 70}{190}$

82. $x^6 - 1 = 0$

83. $x^2 - 2px - q = 0$

84. $(x + 3a)(x + 4a) = 72a^2$

85. $x^{-1} + x^{-\frac{1}{2}} = 6$

86. $\sqrt{x} + \sqrt{(a - x)} = \sqrt{b}$

87. $3x^2 - 9b^2 = 24ax$

88. $x^6 - x^3 = a$

89. $\sqrt[3]{(x^3 - a^3)} = x - b$

90. $\frac{1}{a + b + x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{x}$

91. $mqx^2 - mnx + pqx = np.$

92. $(3a^2 + b^2)(x^2 - x + 1) = (3b^2 + a^2)(x^2 + x + 1).$

93. $\dfrac{5x(x + 1)}{21} - \dfrac{2x^2 + x - 1}{7} = \dfrac{4(x + 1)}{35}.$

94. $\dfrac{18 + x}{6(3 - x)} + \dfrac{65}{4(3 - x)} = \dfrac{20x + 9}{19 - 7x}.$

95. $(x - \frac{1}{2})(x - \frac{1}{3}) + (x - \frac{1}{3})(x - \frac{1}{4}) = (x - \frac{1}{4})(x - \frac{1}{5}).$

96. $x^2 + 13 + \sqrt{(x^2 + 13)} = 56.$

97. $x^2 - 2x + 6\sqrt{(x^2 - 2x + 5)} = 11.$

98. $\dfrac{x^2}{8a} + \dfrac{2x}{3} = \sqrt{\left(\dfrac{x^3}{3a} + \dfrac{x^2}{4}\right)} - \dfrac{a}{2}.$

99. $1 + \dfrac{1}{\sqrt{(x + 1)}} = \sqrt{\dfrac{x^3}{x^2 - 1}}.$

100. $\overline{(x - 2)^2 - x^2} - (x - 2)^2 - x = 90.$

101. $a + x + \sqrt{(2ax + x^2)} = \sqrt{(ax - x^2)} + \sqrt{(2a^2 - ax - x^2)}.$

102. $(x + 1)^5 + (x - 1)^5 = 19\,\{(x + 1)^3 + (x - 1)^3\}.$

103. $\dfrac{(27a + 8x)^{\frac{2}{3}}}{15x^{\frac{13}{15}}} + \dfrac{8x^{\frac{2}{15}}}{3\sqrt[3]{(27a + 8x)}} = \dfrac{8}{5\sqrt[5]{x}}.$

104. $\dfrac{5}{x - 1} + \dfrac{4}{x + 2} + \dfrac{21}{x - 3} = \dfrac{5}{x + 1} + \dfrac{4}{x - 2} + \dfrac{21}{x + 3}.$

105. $x^2 - x + 5\sqrt{(2x^2 - 5x + 6)} = \frac{1}{2}(3x + 33).$

106. $\sqrt{\left(x - \dfrac{1}{x}\right)} + \sqrt{\left(1 - \dfrac{1}{x}\right)} = x.$

107. $\sqrt[n]{(1 + x)^2} - \sqrt[n]{(1 - x)^2} = \sqrt[n]{(1 - x^2)}.$

108. $\dfrac{1}{x - \sqrt{(2 - x^2)}} - \dfrac{1}{x + \sqrt{(2 - x^2)}} = 1.$

109. $x - 2\sqrt{(x + 2)} = 1 + \sqrt[4]{(x^3 - 3x + 2)}.$

110. $\sqrt[2pq]{x^{p+q}} - \dfrac{1}{2} \cdot \dfrac{a^2 - b^2}{a^2 + b^2}\left(\sqrt[p]{x} + \sqrt[q]{x}\right)^4 = 0.$

XV. SIMULTANEOUS QUADRATIC EQUATIONS.

147. The following form is the most general under which quadratics with two unknown quantities can appear.

$$ax^2 + bxy + cy^2 + dx + ey = f,$$
$$a'x + b'xy + c'y^2 + d'x + e'y = f'.$$

Now, if we eliminate x^2 from these equations, and deduce an expression for x, that expression will contain y^2 and y; and, therefore, when this expression is substituted in either of the original equations, the resulting equation will contain y^4, y^2, and y; hence, the solution of the two given equations implies the solution of an equation of the fourth degree. But when the first powers of the unknowns are wanting, and the equations have the form

$$ax^2 + bxy + cy^2 = d,$$
$$a'x^2 + b'xy + c'y^2 = d',$$ *

they can be solved as equations of the second degree. Besides these, there are equations of various forms which can be solved by applying the general rules of elimination already laid down. There is not, however, any general method of solution which will apply to all equations of this class; their solution is effected by adopting various artifices of analysis, which will be best illustrated by particular examples.

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* Such equations are called *homogeneous*, each term on the left side being of two dimensions with respect to the unknown quantities (Art. 17). It is worthy of remark, that if equations be given homogeneous, every step in the process of solution, if it be correct, will give a homogeneous equation; and thus a simple test is afforded of the accuracy of the various operations.

EXAMPLES.

1. Equations of the form

$$ax^2 + bxy + cy^2 = d,$$
$$a'x^2 + b'xy + c'y^2 = d',$$

in which the unknowns are similarly involved, will be best resolved by putting $x = zy$, deducing two expressions for y^2, equating them, and clearing of fractions; in this way we get

$$a'dz^2 + b'dz + c'd = ad'z^2 + bd'z + cd',$$

or
$$(a'd - ad')z^2 + (b'd - bd')z = cd' - c'd.$$

This quadratic will give z; y and x can then be determined.

2. Equations of the form

$$x^n + y^n = a \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$xy = b \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

may be solved as follows:

From the square of (1), subtract four times the nth power of (2), extract the square root of the remainder, and there is obtained

$$x^n - y^n = \sqrt{(a^2 - 4b^n)};$$

combining this with equation (1), we find

$$x = [\tfrac{1}{2}a + \tfrac{1}{2}\sqrt{(a^2 - 4b^n)}]^{\frac{1}{n}}.$$
$$y = [\tfrac{1}{2}a - \tfrac{1}{2}\sqrt{(a^2 - 4b^n)}]^{\frac{1}{n}}.$$

3. Equations such as

$$x + y = a \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$x^3 + y^3 = b \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

may be resolved by either of the following methods:

From the cube of (1) subtract (2), and there remains

$$3x^2y + 3xy^2 = a^3 - b,$$

or $$3(x+y)xy = 3axy = a^3 - b\,;$$

hence, $$x = \frac{a^3 - b}{3ay}\,:$$

and by substitution, transposition, and division by $3a$,

$$y^2 - ay = \frac{b - a^3}{3a}\,;$$

therefore (Art. 144),

$$y = \frac{a}{2} \pm \sqrt{\frac{4b - a^3}{12a}},$$

hence, $$x = \frac{a}{2} \mp \sqrt{\frac{4b - a^3}{12a}}.$$

Otherwise put $x = v + z$, $y = v - z$; then, substituting these in (2), we find

$$z^2 = \frac{b - 2v^3}{6v},$$

and $$z = \pm\sqrt{\frac{b - 2v^3}{6v}}\,;$$

therefore,

$$x = v \pm \sqrt{\frac{b - 2v^3}{6v}},\ y = v \mp \sqrt{\frac{b - 2v^3}{6v}}.$$

Now, $$x + y = 2v = a\,;$$

therefore, $$v = \frac{a}{2}\,:$$

hence we have

$$x = \frac{a}{2} \pm \sqrt{\frac{b - \frac{a^3}{4}}{3a}} = \frac{a}{2} \pm \sqrt{\frac{4b - a^3}{12a}},$$

and $$y = \frac{a}{2} \mp \sqrt{\frac{4b - a^3}{12a}}.$$

This mode of solution is not applicable to powers above the fifth, as a cubic equation would be produced.

4. Given, $$x + y : x - y :: 13 : 5,$$
$$y^2 + x = 25.$$

By a well known property of proportionals,

$$2x : 2y :: 18 : 8,$$

or

$$x : y :: 9 : 4;$$

therefore,

$$4x = 9y, \text{ and } x = \frac{9y}{4}.$$

Substituting this in the second equation, we have

$$y^2 + \frac{9y}{4} = 25;$$

whence (Art. 144),

$$y = 4, \text{ or } -6\tfrac{1}{4}; \quad x = 9, \text{ or } -14\tfrac{1}{16}.$$

5. Given,

$$\frac{10x + y}{xy} = 3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$9x - 9y = -18 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1), $10x + y = 3xy$,
and from (2), $y = x + 2$;
then, by substitution and transposition, we have

$$3x^2 - 5x = 2;$$

whence (Art. 145),

$$x = 2, \text{ or } -\tfrac{1}{3}; \quad y = 4, \text{ or } \tfrac{5}{3}.$$

6. Given,

$$x^4 - 2x^2y + y^2 = 49 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$x^4 - 2x^2y^2 + y^4 - x^2 + y^2 - 20 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equation (2) may be put under the form

$$(x^2 - y^2)^2 - (x^2 - y^2) = 20.$$

Solving this quadratic, we find

$$x^2 - y^2 = 5, \text{ or } -4;$$

also from (1), $x^2 - y = \pm 7$:
by subtraction,

$$y^2 - y = 2, \text{ or } -12; \ 11, \text{ or } -3.$$

Solving this quadratic for these four values, we find

$$y = 2, \text{ or } -1; \qquad x = \pm 3, \text{ or } \pm \sqrt{6}.$$

$$y = \frac{1 \pm \sqrt{-47}}{2};\ x = \pm \sqrt{\frac{-13 \pm \sqrt{-47}}{2}}.$$

$$y = \frac{1 \pm 3\sqrt{5}}{2};\ x = \pm \sqrt{\frac{15 \pm 3\sqrt{5}}{2}}.$$

$$y = \frac{1 \pm \sqrt{-11}}{2};\ x = \pm \sqrt{\frac{-13 \pm \sqrt{-11}}{2}}.$$

7. Given, $x^2 + y^2 + x + y = 18$ . . . . (1)

$2xy = 12$ . . . . (2)

Adding the two equations, and solving the quadratic which results, we find

$x + y = 5$, or -6; . . . . . . (3)

hence, from (1) $x^2 + y^2 = 13$, or 24;

from this taking (2), and evolving, we have

$x - y = \pm 1$, or $\pm 2\sqrt{3}$; . . . (4)

combining (4) with (3), we find

$x = 3$, or 2, or $-3 \pm \sqrt{3}$;

and $y = 2$, or 3, or $-3 \mp \sqrt{3}$.

8. Given, $xy + xy^2 = 12$,

$x + xy^2 = 18$.

Equating the expressions for x, deduced from these two equations, dividing by $\frac{6}{1+y}$, and then reducing and transposing, we find $2y^2 - 5y = -2$;

whence, $y = 2$, or $\frac{1}{2}$;

and therefore, $x = 2$, or 16.

9. Given $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 3x$ . . . . (1)

$x^{\frac{1}{3}} + y^{\frac{1}{3}} = x$ . . . . (2)

From the square of (2) subtract (1), and in the remainder substitute for $y^{\frac{1}{3}}$ an expression deduced from (2), and by transposing and dividing, we shall obtain

$$x - x^{\frac{1}{3}} = 2;$$

which gives $x = 4$, or 1;
and therefore, $y = 8$, or 0.

10. Given, $\sqrt{y} + \sqrt{x} : \sqrt{y} - \sqrt{x} :: \sqrt{x} + 2 : 1, \quad (1)$

$$\frac{\sqrt{y} + 2}{\sqrt{x}} - 1 = \frac{3\sqrt{x} + 1 + \dfrac{\sqrt{y}}{\sqrt{x}}}{\sqrt{y}} \quad . \quad . \quad . \quad (2)$$

From (1) $\sqrt{y} : \sqrt{x} :: \sqrt{x} + 3 : \sqrt{x} + 1$;
therefore, $\sqrt{xy} + \sqrt{y} = x + 3\sqrt{x} \quad . \quad . \quad . \quad . \quad . \quad (3)$
and from (2),

$$y + 2\sqrt{y} - \sqrt{xy} = 3x + \sqrt{x} + \sqrt{y};$$

by transposition,

$$y + \sqrt{y} - \sqrt{xy} = 3x + \sqrt{x}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The addition of (3) and (4) gives a quadratic; solving it, we have $\sqrt{y} + 1 = \pm(2\sqrt{x} + 1)$;
whence, $\sqrt{y} = 2\sqrt{x}$, and $\sqrt{y} = -2\sqrt{x} - 2$.

From the first of these values, and from (3), we find

$$x = \sqrt{x};$$

whence $x = 1$,
and therefore $y = 4$.

From the second value, and (1), we obtain

$$-\sqrt{x} - 2 : -3\sqrt{x} - 2 :: \sqrt{x} + 2 : 1,$$

or $\sqrt{x} + 2 : 3\sqrt{x} + 2 :: \sqrt{x} + 2 : 1$;
hence, $3\sqrt{x} + 2 = 1$,
and therefore $x = \frac{1}{9}$;
whence $\sqrt{y} = -\frac{8}{3}$,
and $y = \frac{64}{9}$.

11. Given $x + y = s$, and $xy = p$, to find the values of $x^2 + y^2$, $x^3 + y^3$, $x^4 + y^4$, &c.

From the square of the first equation subtract twice the second, and there is obtained $x^2 + y^2 = s^2 - 2p$.

Now $(x+y)(x^2+y^2)=(s^2-2p)s$,

and $xy(x+y)=ps$;

therefore, $x^3+y^3=s^3-3ps$.

Again, $(x+y)(x^3+y^3)=s^4-3ps^2$,

and $xy(x^2+y^2)=ps^2-2p^2$;

therefore, $x^4+y^4=s^4-4ps^2+2p^2$.

In the same manner we should have,

$$x^5+y^5=s^5-5ps^3+5p^2s,$$

a formula which we may employ, as well as the method of example 3, in solving such equations as those of Ex. 26.

The general formula would be found, by a continuation of the process, to be

$$x^n+y^n=s^n-nps^{n-2}+\frac{n(n-3)}{2}p^2s^{n-4}-\frac{n(n-4)(n-5)}{2\,.\,3}p^3s^{n-6}+\&c.$$

EXERCISES.

1. $x+y=3x-3y$,
   $x^3-y^3=56$.
2. $xy+y^2=126$,
   $5y=2x$.
3. $x^2+y^2 : x^2-y^2 :: 17 : 8$,
   $xy^2=45$.
4. $x-y : \sqrt{x}-\sqrt{y} :: 8 : 1$,
   $\sqrt{xy}=15$.
5. $\dfrac{x^2+xy+y^2}{x+y}=7$,
   $\dfrac{x^2-xy+y^2}{x-y}=9$.
6. $4xy=96-x^2y^2$,
   $x+y=6$.
7. $\dfrac{x^2}{y^2}+\dfrac{4x}{y}=\frac{85}{9}$,
   $x-y=2$.
8. $x-\dfrac{x-y}{2}=4$,
   $y-\dfrac{x+3y}{x+2}=1$.
9. $x+4y=14$,
   $4x-2y+y^2=11$.
10. $x+y=a$,
    $xy=b$.
11. $x+y=a$,
    $x^2+y^2=c$.

12. $6x^2 - 5xy + 2y^2 = 12,$
$3x^2 + 2xy - 3y^2 = -3.$

13. $x^2 + xy = 12,$
$xy - 2y^2 = 1.$

14. $(x+y)x = 144,$
$(x-y)y = 14.$

15. $x^3 + y^3 = axy,$
$x + y = 2b.$

16. $x^4 + y^4 = 97,$
$xy = 6.$

17. $x + y = 20,$
$x^3 + y^3 = 2060.$

18. $x^2 + y^2 = 306,$
$x + y = 24.$

19. $x + y = 7,$
$x^4 + y^4 = 641.$

20. $x^2 + y^2 = 85,$
$xy = 42.$

21. $x^3 + y^3 = 152,$
$x^2y + xy^2 = 120.$

22. $x + y + z = 21,$
$x^2 + y^2 + z^2 = 189,$
$y^2 = xz.$

23. $x^5 + y^5 = 17050,$
$x + y = 10.$

24. $x^{x+y} = y^{4a},$
$y^{x+y} = x^a.$

25. $2x^2 - 3xy + 3y^2 = 5,$
$3x^2 + xy - 2y^2 = 12.$

26. $x + y = a,$
$x^5 + y^5 = b.$

27. $x + y = a,$
$x^4 + y^4 = b.$

28. $x^2 + y^2 = 65,$
$xy = 28.$

29. $x + y = 18,$
$x^3 + y^3 = 4914.$

30. $x^5 - y^5 = 3093,$
$x - y = 3.$

31. $x - 2\sqrt{xy} + y - \sqrt{x} + \sqrt{y} = 0,$
$\sqrt{x} + \sqrt{y} = 5.$

32. $(x^2 + y^2)(x^3 + y^3) = 280,$
$x + y = 4.$

33. $x + y - \dfrac{\sqrt{(x+y)}}{\sqrt{(x-y)}} = \dfrac{6}{x-y},$
$x^2 + y^2 = 41.$

34. $\dfrac{\sqrt{(y^2+1)}+1}{y} = \dfrac{\sqrt{(x+9)}+3}{\sqrt{x}}.$

$x(y+1)^2 = 36\left(y^2 + \dfrac{16}{9}\right).$

35. $2y^2 - 4xy + 3x^2 = 17,$
$y^2 - x^2 = 16.$

XVI. THEORY OF QUADRATIC EQUATIONS.

148. Any number which, being substituted for the unknown quantity in an equation, renders its two sides identical, is called a *root* of the equation.

149. We have seen in Art. 144 that one general form of a quadratic equation is

$$x^2 + px = q, \text{ or } x^2 + px - q = 0.$$

Now, if r be a root of this equation, that is, a value of x, we shall have by substitution,

$$r^2 + pr = q;$$

whence, $$x^2 + px = r^2 + pr,$$

or, $$x^2 - r^2 + px - pr = 0;$$

hence, $$(x + r)(x - r) + p(x - r) = 0;$$

that is, $$(x - r)(x + r + p) = 0:$$

from which last equation it follows, that, *if* r *be a root of the equation* $x^2 + px = q$, *the left member is divisible by* $x - r$; and conversely, *if* $x - r$ *divide the left member without remainder,* r *is a root of the equation.*

150. Since a product becomes zero when either of its factors becomes zero, $(x - r)(x + r + p) = 0$ both when $x - r = 0$, which gives $x = r$; and when $x + r + p = 0$, which gives $x = -r - p$; hence, r being one root, $-r - p$ is the other; and *hence a quadratic has two roots.*

151. Since $x^2 + px - q = (x - r)(x + r + p)$, it follows that *a quadratic equation is the product of two factors, which are of the first degree with respect to* x.

152. Adding together the two roots we have

$$r + (-r - p) = -p;$$

hence, *the sum of the roots of a quadratic equation is equal to the coefficient of the first power of the unknown with its sign changed.*

153. Since $r^2 + pr - q = 0$, transposing and changing all the signs we have

$$-r^2 - pr = -q, \text{ or } r(-r-p) = -q\,;$$

and, therefore, *the product of the roots of a quadratic equation is equal to the absolute term with its sign changed.*[*]

154. The general value of x (Art. 145),

$$x = \frac{\mp b \pm \sqrt{(b^2 \pm 4ac)}}{2a}$$

includes the four following forms:

$$\left.\begin{matrix}1.\\2.\end{matrix}\right\}ax^2 \pm bx = +c \quad .\ .\ .\ .\ . \quad x = \begin{cases} \dfrac{-b \pm \sqrt{(b^2 + 4ac)}}{2a}, \\ \text{or} \\ \dfrac{b \pm \sqrt{(b^2 + 4ac)}}{2a}. \end{cases}$$

$$\left.\begin{matrix}3.\\4.\end{matrix}\right\}ax^2 \pm bx = -c \quad .\ .\ .\ .\ . \quad x = \begin{cases} \dfrac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}, \\ \text{or} \\ \dfrac{b \pm \sqrt{(b^2 - 4ac)}}{2a}. \end{cases}$$

In the four values of the first two of these forms, since the quantities under the radical sign are both positive, the square root can be extracted either exactly or approximately, and therefore the roots of the equation are real; and since

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[*] We might also deduce these properties immediately from the general values in Art. 144. The work is left as an exercise for the student.

$\sqrt{(b^2 + 4ac)}$ is necessarily a greater number than b, the roots will be positive when the sign of the radical is +, and negative when the sign of the radical is −.

Again, in the four values of the last two of these forms, in order to render the evolution possible, and the roots real, we must have $b^2 > 4ac$; but if $b^2 < 4ac$, the roots are both imaginary; and since $\sqrt{(b^2 - 4ac)}$ is a less number than b, the roots will be negative when b is positive in the equation, and positive when b is negative.

If we suppose $b^2 = 4ac$, or $c = \frac{b^2}{4a}$, in the equation $ax^2 \pm bx = -c$, the radical quantity vanishes, and the two values of x are, in the one case $-\frac{b}{2a}$, and in the other $\frac{b}{2a}$; the roots are then equal to one another, and the equation is the square of a binomial.

155. We might deduce these results from the principles established in Arts. 152, 153; for if r, r', are the roots of the equation $x^2 + px = q$, we have

$$r + r' = -p, \; rr' = -q;$$

or

$$p = -(r + r'), \text{ and } q = -rr';$$

the roots have therefore different signs, since their product is negative. Also, since the sum of the roots is negative when p is positive, and positive when p is negative (Art. 151), it follows that the root which is the greater number has a different sign from p.

Again, in the equation $x^2 \pm px = -q$, q being negative, the two roots will have the same sign, since their product is positive, or has an opposite sign to the absolute term (Art. 153); but as their sum has always a different sign from p, they are, under these two forms, both negative when p is positive, and both positive when p is negative. But under the two forms $x^2 \mp px = q$, q being positive, the roots will have opposite signs; that which is the greater number being positive when p is negative, and negative when p is positive.

These results, as regards the signs of the roots will be more easily remembered, if the equations are presented under the form in which all the quantities are transposed to one side. These forms and the characters of the roots are as follows :—

Equations.	Character of roots.
1. $x^2 + px + q$	Both negative.
2. $x^2 - px + q$	Both positive.
3. $x^2 - px - q$	Greater, positive; less, negative.
4. $x^2 + px - q$	Greater, negative; less, positive.

Equations.	Roots.	
1. $x^2 + 16x + 48 = 0$	-12, and -4	Both $-$
2. $x^2 - 16x + 48 = 0$	$+12$, and $+4$	Both $+$
3. $x^2 - 8x - 48 = 0$	$+12$, and -4	Gr., $+$; less, $-$
4. $x^2 + 8x - 48 = 0$	-12, and $+4$	Gr., $-$; less, $+$

The absolute term has under these forms the *same* sign as the product of the roots; while that of the coefficient p, not altered by transposition, has still a sign opposite to that of the sum of the roots.

156. If we compare the results to which we have arrived in the last Article with the signs of the terms of the general equation, we shall perceive that there is always a connection between the number of positive roots, and the number of changes of the signs from + to −, or from − to +, Thus, in the equation $x^2 + px + q = 0$, since the root that is the greater number is always of a different sign from p, and since in this case the product of the roots is positive, the other root also must be negative; thus, *there is no positive root, and no change in the signs of the terms of the equation.*

Again, in the equation $x^2 - px + q = 0$, the root that is the greater number must be positive; but the product is positive, and therefore also the other root; there are then, two positive roots, and two changes of sign in the equation. And in the same manner it may be shewn, that of the equations $x^2 + px - q = 0$, and $x^2 - px - q = 0$, having but one change of sign, *there is one root positive, and the other negative.*

157. From the general principles established in the preceding Articles, it follows,

1st. That when a quadratic is reduced to the form $x^2 \pm px = \pm q$, its roots may be found by inspection, whenever they are such as can be determined in finite terms.

2d. That when the roots of a quadratic are given, the equation will be formed by putting the difference between the unknown quantity and these roots equal to zero, transposing all the terms to one side, and finding the product of the results.

3d. That the factors of a quadratic expression will be found by putting it equal to zero, and solving the equation that results.

EXAMPLES.

1. Find, by inspection, the roots of the equation

$$x^2 - 7x = 44, \text{ or } x^2 - 7x - 44 = 0.$$

Here it is required to find two numbers, whose product shall be -44, and sum $+7$; the numbers plainly are 11 and -4. We know from the last Art. also, that one root only is positive; and from Art. 154, that that is the greater root.

2. Required the quadratic equation whose roots are 5 and -7.

Here $x = 5$, $x = -7$; therefore, transposing, we have $x - 5 = 0$, and $x + 7 = 0$: multiplying, we find $x^2 + 2x - 35 = 0$, or $x^2 + 2x = 35$; which is the required quadratic.

3. Find the factors of $2x - 15 + x^2 + a(3 - x)$.

The roots are here found to be $a - 5$, and 3; hence the factors are $x - a + 5$, and $x - 3$.

EXERCISES.

1. Required the factors of $pq^2 + \dfrac{m^2 - n^2}{p} - 2mq$.
2. Required the factors of $(a + b)x + x^2 - 9x + (b - 4)a + (4 - b)5$.
3. Required the factors of the expression

$$\frac{a}{a+2} - \frac{a-9}{3a-20} - \frac{9}{13}.$$

4. Required the factors of $x^2 - 5x - 6000$.
5. Required the quadratic whose roots are 10 and -9.
6. Required the quadratic whose roots are $\frac{1}{2}$ and $\frac{1}{3}$.
7. Required the factors of $a^2 + 3a - 28$.
8. Required the factors of $2a^2c - 3cx^2 - b^2c - acx + bcx^2 - a^2c$.
9. Required the quadratic whose roots are 7 and $-\frac{11}{12}$.
10. Required the factors of $3400 + 111x - x^2$.
11. Solve by inspection the equation $x^2 = 18x - 77$.
12. Solve by inspection the equation $110x^2 - 21x + 1 = 0$.
13. Solve the equation $x^2 - 16x = 36$ by inspection.
14. Solve the equation $x^2 + 4x = 320$ by inspection.
15. Shew that in the equations $2x - \frac{1}{4}x^2 = 10$; $3x^2 \pm 8x = -6$; $2x^2 + 15 = 3x$, the roots are imaginary.
16. Shew that in the equation $2x^2 - 31x = -15$, the roots are real.

XVII. PROBLEMS PRODUCING EQUATIONS

OF THE

FIRST AND SECOND DEGREES.

158. The object of an algebraical problem is to find the value of some unknown quantity, by means of the relations which it bears to others that are known: and since equations have the same object (Art. 122), it is plain that problems may be expressed by equations. In order to effect this, the data of the problem, or the known quantities, and the required or unknown quantities, are denoted as is explained in Art. 122; and the same operations are then performed on these quantities, as are described by the conditions of the problem. It is, of course, assumed, that these conditions must be such as to establish a mutual dependence between the known and unknown quantities, and such as to be expressible by means of the ordinary operations of Algebra. With respect to the number of equations which the conditions will give, it is sufficient to refer to Art. 129.

The chief difficulty in the solution of a problem consists in adapting the operations to be performed to the conditions given by the problem; that is, in the reduction of the problem to algebraical language. For this, however, no fixed Rule can be laid down; and it must therefore be learned by practice.*

When it happens, from the nature of the question, that fractional parts of an unknown are to be taken, it will be better to represent the unknown by such a multiple of x, y, &c., as will avoid fractions. The answers which are obtained should be verified by trying if they will fulfil the conditions of the problem.

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* Considerable difficulty is often felt in the first attempts to translate the conditions of a question into algebraic language. To remove this difficulty, the Miscellaneous Exercises, pp. 49, 50, have been given, chiefly by way of preparation for this section.

EXAMPLES.

1. What number is that, whose half exceeds its third part as much as 7 exceeds the number itself?

Let x represent the number.

Then the excess of the half above the third of the number is $\frac{x}{2}-\frac{x}{3}$; and the excess of 7 above the number itself is $7-x$; therefore, by the conditions of the question, we have

$$\frac{x}{2}-\frac{x}{3}=7-x,$$

which gives $x=6$.

To avoid fractions, we might take $6x$ for the number; we should then have $3x-2x=7-6x$;
whence, $x=1$, and $6x=6$;
the number required.

2. Of a bequest of £4755, B got three times as much as A, C as much as A and B together, and D as much as C and B. Required the share of each.

Here, the other shares being expressed in terms of A's, let x represent A's share; then B's is $3x$, C's is $x+3x$, and D's is $x+3x+3x$; then, by the question, we have

$$x+3x+x+3x+x+3x+3x=4755;$$

which gives A's share £317; B's, £951; C's, £1268; D's £2219.

3. If the down-train from London travel 26 miles per hour; and the up-train from Liverpool, 24 miles per hour; in what time will they meet, and what distance will each have travelled, the distance by the railway being supposed to be 200 miles.

Let the number of hours required be x. Then $26x$ and $24x$ are the respective distances travelled; therefore, according to the data, $$26x + 24x = 200;$$
whence, $$x = 4.$$

They meet in 4 hours; and the distances passed over are 104 miles, and 96 miles.

4. If A can do a piece of work in 6 days, and B in 8 days; in how many days will they do it if they both work at it together?

Let x be the number of days. Then, in one day, A can do $\frac{1}{6}$ of the work; and B, $\frac{1}{8}$: therefore, in x days, they can do x times $\frac{1}{6}$ and x times $\frac{1}{8}$ of the work.

Therefore, representing the whole work by 1, we have the equation $$\frac{x}{6} + \frac{x}{8} = 1;$$
whence, $$x = \frac{48}{14};$$
and therefore, $$x = 3\tfrac{3}{7}.$$

5. How many gallons of spirits, at 9 shillings per gallon, must be mixed with 20, at 13 shillings, so that the mixture may be worth 10 shillings per gallon?

Let x be the quantity at 9 shillings. Then $9x$ shillings is its price.

Also, the whole quantity is $x + 20$, and its price is $10(x + 20)$ shillings; and 260 shillings is the price of 20 at 13 shillings. Then

$$9x + 260 = 10x + 200\,;$$

therefore, $$x = 60\,;$$

and the whole quantity is 80.

6. A person being asked the hour, answered that it was between 5 and 6 o'clock, and that the hour and minute hands were exactly together. What is the time?

Let $x =$ the time past 5. Then, as the minute hand makes 12 revolutions for one of the hour hand, we have

$$12 : 1 :: 5 + x : x\,;$$

whence, the time is easily found to be $27'\ 16\frac{4}{11}''$ past 5 o'clock.

7. A general, ranging a division of his army in the form of a solid square, finds he has 34 men to spare; but, increasing the side by one man, he wants 59 to fill up the square. How many men had he?

Let x and $x + 1$ be the numbers of men in the sides of the two squares: then $x^2 + 34$, and $x^2 + 2x + 1 - 59$, are the numbers of men in each. Hence, we easily find the number of men to be 2150; and the numbers in the two sides, 46 and 47.

8. There is a number consisting of two digits, which is equal to four times the sum of those digits; and if 9 be subtracted from twice the number, the digits will be inverted. Required the number.

Let the digits be x and y; then the number is $10x + y$, and the number inverted is $10y + x$. Therefore, by the

question $10x + y = 4(x + y)$,
and $2(10x + y) - 9 = 10y + x$;
from which two equations we find the number to be 36.

9. A merchant sold a quantity of brandy for £39, and gained as much per cent as the brandy cost him. Required the price of the brandy.

Let $x =$ the price of the brandy; then £39 $- x$ is the gain. Also, $39 - x : x :: x : 100$;
whence, $x^2 = 3900 - 100x$,
or $x^2 + 100x = 3900$.

Solving this quadratic, we find

$$x = 30, \text{ or } - 130;$$

the price is therefore £30.*

10. A rectangular bowling-green is of a certain size; and if it were 5 feet broader, and 4 feet longer, it would contain 116 feet more; but if it were four feet broader, and 5 feet longer, it would contain 113 feet more. Required its length and breadth.

Let $x =$ the length, and $y =$ the breadth; then the area $= xy$. Also $(x + 4) \times (y + 5) = xy + 116$,
and $(x + 5) \times (y + 4) = xy + 113$;
from which we find $x = 12, y = 9$.

---

* This negative value shews (Art. 5) that the question must be modified. It will be as follows:

"A merchant sold a quantity of brandy, by which he *lost* £39 *more* than the first cost, and found that his loss was as much per cent as the brandy cost him. Required the price."

The equation will now be

$$x^2 = 3900 + 100x;$$

and the values of x will be 130, and — 30: the price is therefore £130. The value — 30 shows that *this* question must be modified; when the conditions are changed, it will, of course, be as in the text.

11. A vintner sold 7 dozen of sherry, and 12 dozen of claret, for £50; and finds that he has sold 3 dozen more of sherry for £10, than he has of claret for £6. Required the price of each.

Let $x =$ the price of a dozen of sherry in pounds; then $\frac{10}{x} =$ number of dozen of sherry for £10, and $\frac{10}{x} - 3 =$ number of dozen of claret for £6; therefore, $6 \div \left(\frac{10}{x} - 3\right) =$ the price of a dozen of claret. Hence, $7x + \frac{72x}{10 - 3x} = 50$. From this we find that the sherry was £2, and the claret £3.

This might also be solved as follows: Let the prices be x and y: then, by the question

$$7x + 12y = 50,$$

and
$$\frac{10}{x} - \frac{6}{y} = 3.$$

Deducing two expressions for x from these two equations, equating them, and solving the resulting quadratic, we find $y =$ £3; and therefore, from the first equation, $x =$ £2.

12. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel, in going 120 yards; but if the circumference of each wheel be increased one yard, it will make only 4 revolutions more than the hind-wheel in the same space. Required the circumference of each.

Let x and y be the number of yards in the circumferences of the wheels; then $\frac{120}{x}$ and $\frac{120}{x+1}$, $\frac{120}{y}$ and $\frac{120}{y+1}$, express the number of revolutions of the wheels: we have therefore $\frac{120}{x} = \frac{120}{y} - 6$, and $\frac{120}{x+1} = \frac{120}{y+1} - 4$. From the first of these, we find $xy = 20x - 20y$; from the second, $xy = 29x - 1 - 31y$. Equating these two, deducing an expression for x, and substituting it in the equation $xy = 20(x - y)$, a quadratic is obtained. Solving it, we find $y = 4$, and therefore $x = 5$.

EXERCISES.*

1. What number is that whose half and third part together are equal to 20?

2. What is that number whose quadruple exceeds its half by 14?

3. What number is that, the double of which increased by b, is equal to a?

4. What number is that from whose double if a be taken, the remainder will be b?

5. What number is that, $\frac{1}{a}$ and $\frac{1}{b}$ of which together are equal to c?

6. Find two numbers differing by 9, and such that four times the less may exceed twice the greater by 14.

7. Divide 56 into two parts, having the ratio of 11 to 17.

8. Divide £150 between two persons, giving the one £14 more than the other.

9. What is that number, from whose treble if 4 be subtracted, three-fifths of the remainder will be 12?

10. Divide £300 among three persons, giving the second £9 more than the third, and the first £15 more than the second.

Solve this question in general terms, by putting $300 = c$, $9 = a$, $15 = b$.

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* The questions which are solved in general terms, should be illustrated by examples with particular numbers.

By means of these solve such questions as the following :

1. A cask which holds 100 gallons is filled with a mixture of wine, water, and spirits ; there are 15 gallons more of wine than of water, and 25 gallons of spirits more than of wine. Find the quantity of each.

2. Three workmen, A, B, C, were cutting a drain 159 perches long, of which B cut 15 perches more than A, and C 24 perches more than B. How much did each cut ?

11. Required a number, the half of which is as much less than 100, as its double exceeds 100.

The following is the general statement of this question :

Find a quantity x, such that mx may be as much less than a, as nx is greater than b.

12. A post is one-third in mud, one-fourth in water, and 10 feet above water. Required its length.

13. One-tenth of a pole is coloured red, one-twentieth orange, one-thirtieth yellow, one-fortieth green, one-fiftieth blue, one-sixtieth indigo, and the remainder, which is 302 inches long, violet. Required its length.

14. A fish was caught whose tail weighed 9lbs., its head as much as its tail and half its body, and its body as much as its head and tail. Required its whole weight.

15. If the down-train from London travels 36 miles per hour, and the up-train from Edinburgh 30 miles per hour, in how many hours will they meet, and how far from London, the distance being 396 miles ?

16. Suppose every thing to be as in the last question, except that the down-train starts an hour earlier than the other ; where will they meet ?

17. The sum of the three sides of a triangle is 75 ; the base is 11 feet longer than one side, and 16 feet longer than the other. Required the three sides.

18. Find two numbers such that their sum shall be a, and the one m times as great as the other.

19. Divide a into three such parts that the second may be m times, and the third n times as great as the first.

This is the general statement of such questions as the following :—

In a company of 45 persons, there are three times as many English, and five times as many Scotch as Irish. Required the number of each nation.

20. At a certain election 812 persons voted, and the successful candidate had a majority of 42 votes. Find how many voted for each candidate.

Generalize this by supposing that a persons voted, and that the majority was b votes.

21. A man, at the time he was married, was three times as old as his wife; but after they had lived together 15 years, he was only twice as old. What were their ages on their wedding day?

22. A cistern which holds 820 gallons, is filled in 20 minutes by three pipes, of which one conveys 10 gallons per minute more, and the other 5 gallons per minute less than the third. How much does each pipe convey in a minute?

23. Having bought a certain quantity of brandy at 19 shillings a gallon, and a quantity of rum exceeding that of the brandy by 9 gallons, at 15 shillings a gallon; I find that I paid one shilling more for the brandy than for the rum. How many gallons were there of each?

24. A man and his wife usually drank out a vessel of beer in 12 days; but when the man was from home, the vessel lasted the woman 30 days. In how many days would the man alone drink it out?

25. It is required to divide 34 into two such parts, that

the difference between the greater and 18 shall be to the difference between 18 and the less, as 2 is to 3.

26. A man is engaged to work for 30 days, on condition of receiving 2s. for every day he worked, and of forfeiting 6d. for every day he was idle; at the end of the time he received 35s. Required the number of days he worked, and the number he was idle.

Generalize this by putting $30 = n$, 24d. $= p$ pence, and 6d. $= q$ pence; and state the answer in terms of these quantities.

27. If A can do a piece of work in a days, and B in b days, in how many days will they finish it if they work at it together?

Find also an expression for the joint time in the case of three persons, A, working a days, B, b days, and C, c days; and in the case of four persons, A working a days, B, b days, C, c days, and D, d days.

From these expressions deduce particular solutions by taking numbers for the letters in the general expressions. Thus suppose A can do the work in 6 days, B in 8, and C in 10, what would be the joint time. Suppose also A to do it in 12 days, and B in 15 days, find the joint time; and that in the case of four persons the times are respectively 4, 5, 6, and 8 days.

28. Some bees were sitting on a tree; at one time there flew away a number of them represented by the square root of half the number in the swarm; and again, eight-ninths of the whole swarm, two only remaining. How many bees were there?

This problem is given in the Hindoo treatise referred to in the note p. 168.

29. The time is between one and two o'clock of the day, and the hour and minute hands are exactly together. What is the exact time?

30. Suppose two traders each to double his capital annually, except an outlay of £100; and that at the end of three years the capital of the one is found to be doubled, while the other has only half as much as he had at first; how much had each to begin with?

31. In 1830, A's age was 50, and B's 35. Required the date at which A's age is double of B's.*

32. A father's age is now treble of his son's; what are their present ages, the age of the father 7 years ago having been five times that of the son?

To find a general solution, put $3 = m$, $5 = n$, and $7 = a$.

33. A person rows a distance of 20 miles upon a river and back again in 10 hours, the stream flowing uniformly in the same direction all the time; and he finds that he can row 2 miles against the stream in the same time that he rows 3 miles with it. Find the velocity of the stream, and the times of going and returning.

34. A composition of copper and tin, containing 100 cubic inches, weighed 505 ounces; how many ounces of each metal did it contain, supposing a cubic inch of copper to weigh $5\frac{1}{4}$ ounces, and one of tin $4\frac{1}{4}$ ounces?

35. From each of 16 coins an artist filed the worth of half a crown, and then offered them in payment for their original value; but being detected, the pieces were found to be worth only 8 guineas. Required their original value.

36. One person says to another, "Give me £100, and then I shall have as much as you." "No," answered the

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* If this question were enunciated as follows, "In 1830, A's age was 50, and B's 35; when will A's age be double of B's;" the answer, $x = -20$, would shew that the date is *before* 1830, and therefore that the words "when will A's age be" are incorrect, and must be changed for the words "required the date at which A's age is." *See* Arts. 5, 6.

other; "but do you give me £100, and I shall have twice as much as you." How much has each?

37. A wine merchant sold at one time 20 dozen of port wine, and 30 of sherry, and for the whole received £120; and again, at the same prices, 30 dozen of port, and 25 of sherry, and for the whole got £140. Required the price of a dozen of each sort.

38. Required a fraction, to the numerator of which if 4 be added, the value is $\frac{1}{2}$; but if 7 be added to the denominator, the value is $\frac{1}{5}$.

Or for a general solution:

Required a fraction, which, if a be added to its numerator, will become b; but if c be added to its denominator, will become d.

39. A bill of £120 was paid in guineas and moidores, and the number of pieces of both sorts used was 100. How many were there of each sort, the guinea being 21, and the moidore 27, shillings?

40. A man and his wife could drink a barrel of beer in 15 days. After drinking together 6 days, the woman alone drank the remainder in 30 days. In what time would either, alone, drink a barrel?

41. With 28 bushels of barley, at 2s. 4d. per bushel, a farmer would mix rye at 3s., and wheat at 4s; so that the whole mixture may consist of 100 bushels, and be worth 3s. 4d. per bushel. How much rye and wheat must he take?

42. A countryman being employed by a poulterer to drive a flock of geese and turkeys to London, in order to distinguish his own from any he might meet on the road, pulled three feathers out of the tail of each turkey, and one out of the tail of each goose; and, upon counting them, found that the number of turkeys' feathers exceeded twice those of the geese by 15. Having bought ten geese, and

sold fifteen turkeys by the way, he was surprised to find, as he drove them into the poulterer's yard, that the number of geese exceeded the number of turkeys in the proportion of 7 to 3. Required the number of each at first.

43. A number consisting of two digits, when divided by 4, gives a certain quotient, and a remainder of 3; when divided by 9, gives another quotient, and a remainder of 8. Now, the value of the digit on the left hand is equal to the quotient which was got when the number was divided by 9; and the other digit is $\frac{1}{17}$th of the quotient got when the number was divided by 4. Required the number.

44. Find a number consisting of two places of figures such that when it is divided by the difference of its digits the quotient is 21; and when it is divided by the sum of its digits and the quotient increased by 17, the digits are inverted.

45. A man loses at a gaming table one-half his money, and then gains 6s.; afterwards, he loses one-third of what remains, and then gains 12s.; lastly, he loses one-fourth of what now remains, and finds that he has only 2 guineas remaining. Find the sum which he had at first.

46. A field is in the form of a rectangular parallelogram, whose length exceeds its breadth by 16 yards; and it contains 960 square yards. Required the length and breadth.

47. There are three numbers, the difference of whose differences is 5; their sum is 44, and continued product 1950. Find the numbers.

48. Having sold goods for £24, I gained as much per cent as the whole cost me. Required the cost.

49. There are two square courts that are paved with stones a foot square each: the side of one court exceeds that of the other by 12 feet; and both their pavements taken together contain 2120 stones. What are their lengths separately?

50. A person bought some sheep for £72, and found, that if he had bought 6 more for the same money, he would have paid £1 less for each. Required the number bought, and the price of each.

51. The difference between the hypotenuse and base of a right-angled triangle is 6; and the difference between the hypotenuse and perpendicular is 3. Required the sides.

52. There is a field in the form of a rectangle, whose length is to its breadth as 6 to 5; one-sixth of the whole area being planted, there remain for ploughing 625 square yards. Required the dimensions of the field.

53. Two messengers were dispatched at the same time to a place 90 miles distant; the former of whom, by riding one mile an hour faster than the other, arrived at the end of his journey an hour before the other. Required the rate of each per hour.

54. A regiment of soldiers, consisting of 1066 men, is formed into two squares, one of which has 4 men more in a side than the other. Required the number in each side.

55. A company at a tavern had £8 : 15s. to pay; but before the bill was paid, two of them went away, and in consequence, those who remained had each 10s. more to pay. How many persons were in the company at first?

56. Find two numbers whose product is 300, and such that if 10 be added to the less, and 8 subtracted from the greater, the product of the sum and remainder shall also be 300.

57. Find two numbers such that if $\frac{2}{3}$ of the less be added to $\frac{1}{4}$ of the greater, the sum will be 7; but if $\frac{1}{3}$ of the greater be taken from the less the remainder will be 2.

58. A person starts to walk at a uniform rate without stopping, from a place A to another place B, and back again; at the same time that another person starts from B to walk in the same way to A and back. They meet a mile and a half

from B; and again, an hour after, a mile from A. Find their rates of walking, and the distance from A to B.

59. The sum of the numerator and denominator of a fraction is equal to 6 times the fraction; and the sum of the numerator and denominator of its reciprocal is equal to $1\frac{1}{2}$ times its reciprocal. What is the fraction?

60. It is required to find the three sides of a right-angled triangle, from the following data; the number of square feet in the area is equal to the number of feet in the hypotenuse, together with the sum of the feet in the other two sides; and the square described upon the hypotenuse is less than the square described upon the sum of the two sides, by half the product of the base and area.

61. A man travelled 105 miles, and found that if he had not travelled so fast by 2 miles an hour, he would have been 6 hours longer in performing the same journey. How many miles did he go per hour?

62. A farmer bought two flocks of sheep for £65 : 13s., one containing 5 more than the other; each sheep cost as many shillings as there were sheep in the flock. Required the number in each.

63. The joint stock of two partners, A and B, was £416; A's money was in trade 9 months, and B's 6 months; when they shared stock and gain, A received £228, and B £252. Required each man's stock.

64. Two persons, A and B, travelled on the same road and at the same rate to London. At the 50th milestone from London A overtook a flock of geese, which travelled at the rate of 3 miles in 2 hours, and 2 hours afterwards he met a stage waggon which travelled at the rate of 9 miles in 4 hours. B overtook the flock of geese at the 45th milestone from London, and met the stage waggon 40 minutes before he came to the 31st milestone. Where was B when A reached London?

65. In comparing the rates of a watch and a clock, it was observed, on one morning, when it was 12 by the clock that the watch was at 59 minutes 49 seconds past 11; and, two mornings after, when it was 9 by the clock, the watch was at 59 minutes 58 seconds past 8. The clock is known to gain $\frac{1}{10}$ of a second in 24 hours. Find the gaining rate of the watch.

66. If A and B together can perform a piece of work in *a* days, A and C together in *b* days, and B and C together in *c* days, find the time in which each can perform the work separately.

Find the joint time by substituting 8 days for *a*, 9 days for *b*, and 10 for *c*.

67. A detachment of soldiers from a regiment being ordered to march on a particular service, each company furnished 4 times as many men as there were companies in the regiment; but these being found to be insufficient, each company furnished 3 more men, when their number was found to be increased in the ratio of 17 to 16. How many companies were in the regiment?

68. A sets off from London to York, and B at the same time from York to London; they travel uniformly; A reaches York 16 hours, and B London 36 hours after they have met on the road. Find in what time each has performed the journey.

69. The mail train on a railway starts a certain time after a luggage train from the same terminus, and the time is so adjusted that before arriving at the other terminus, the trains will just escape collision and no more. Now, it happens that the speed of the luggage train, from an accident to the engine, is suddenly reduced one-half, after performing two-thirds of the journey, and a collision takes place 12 miles from the end of it; the proper speeds of the trains being 36 and 24 miles per hour, find the length of the railway, and the difference in the time of starting.

Generalize the expressions by calling 12 miles, a; and the respective speeds b and b'; b being greater than b'.

70. The distance from Edinburgh to Glasgow is $47\frac{1}{2}$ miles by railway; the express down-train leaves Edinburgh at 10.30 A.M., and reaches Glasgow at 11.40; the up-train leaves Glasgow at 10. A.M., and reaches Edinburgh at 11.20. Both trains perform the journey without stopping at any station, and the speed of each is uniform; find where they will meet.

XVIII. INDETERMINATE COEFFICIENTS.

159. If it were required to resolve the fraction $\frac{a}{a+bx}$ into an infinite series, without actually dividing, we at once see (Art. 81) that the series will consist of the powers of x multiplied by certain coefficients, the indices of the powers of x beginning at zero and increasing by a unit in each successive term. That such development is possible, will also appear from Art. 92, since the given expression is the same as $a(a+bx)^{-1}$. It is not, however, obvious what the coefficients will be; but it is plain that they are independent of x, and are expressed in terms of a and b. Let us then assume the coefficients to be A, B, C, D, &c., and we shall have

$$\frac{a}{a+bx} = 1 + \mathrm{A}x + \mathrm{B}x^2 + \mathrm{C}x^3 + \mathrm{D}x^4, \&c.,$$

an *identical* equation, the assumed series being merely the supposed result of the operation indicated by the left member. To determine these coefficients, multiply both members by the denominator, and arrange the terms by the powers of x; we thus obtain

$$\begin{array}{l|l|l|l|l} a = a + \mathrm{A}a & x + \mathrm{B}a & x^2 + \mathrm{C}a & x^3 + \mathrm{D}a & x^4 \text{ \&c.}^* \\ \qquad\quad + b & \quad + \mathrm{A}b & \quad + \mathrm{B}b & \quad + \mathrm{C}b & \end{array}$$

Now, in order that this equation may be satisfied by its members being rendered identical, the terms containing x must vanish. This will take place, whatever be the value of x, when the coefficients of its powers are equal to zero, that is, when

$\mathrm{A}a + b = 0$, from which we deduce $\mathrm{A} = -\frac{b}{a}$;

$\mathrm{B}a + \mathrm{A}b = 0$ . . . . $\mathrm{B} = -\mathrm{A}\frac{b}{a} = \frac{b^2}{a^2}$;

$\mathrm{C}a + \mathrm{B}b = 0$ . . . . $\mathrm{C} = -\mathrm{B}\frac{b}{a} = -\frac{b^3}{a^3}$;

$\mathrm{D}a + \mathrm{C}b = 0$ . . . . $\mathrm{D} = -\mathrm{C}\frac{b}{a} = \frac{b^4}{a^4}$; &c.

The series will therefore be

$$\frac{a}{a + bx} = 1 - \frac{b}{a}x + \frac{b^2}{a^2}x^2 - \frac{b^3}{a^3}x^3 + \frac{b^4}{a^4}x^4 \text{ \&c.};$$

the same as would be obtained by actual division, which in this case would be more direct.

This indirect method of determining the development of a series, is called the method of *assumed* or *indeterminate coefficients*. It is applied to the same purposes as the direct methods in Arts. 81, 91, 99, and to various others in the higher mathematics.†

The general principle of the method consists in assuming unknown coefficients, and then deducing two expressions

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[*] That the left member is of the same form as the right, will be plain when it is written thus, $a + 0x + 0x^2 + 0x^3$, &c.

[†] *Undetermined coefficients*, or *coefficients to be determined*, would be a better name than *indeterminate*; as that word has, in other parts of Algebra, a different signification.

or series of the same form, which, by the nature of the process by which they are obtained, are to be identical with one another. A series of equations, giving the values of these coefficients, is then obtained by equating the corresponding terms, or their coefficients, and thus rendering the expressions identical.

160. If the series $A + Bx + Cx^2 + Dx^3$ &c. be equal to the series $A' + B'x + C'x^2 + D'x^3$ &c. for all values of x, then will the coefficients of x in the two series be respectively equal to one another. For, equating these two series, and transposing all the terms to the right member, we have

$$0 = A' - A + (B' - B)x + (C' - C)x^2 + (D' - D)x^3 \text{ \&c.};$$

which cannot be true universally, unless

$$A' - A = 0,\ B' - B = 0,\ C' - C = 0,\ D' - D = 0.$$

Hence, $A' = A$, $B' = B$, $C' = C$, $D' = D$, &c.

161. A series with indeterminate coefficients, is generally assumed to proceed according to the ascending integral and positive powers of x, beginning with x^0; but in many series this is not the case: the error in our assumption is then shewn, either by some impossible result, or by those terms which do not exist in the actual series becoming zero in the course of the calculation.

Thus, if it be required to develope $\frac{1}{3x - x^2}$, assuming the same series as before, we should have

$$0 = -1 + 3Ax + (3B - A)x^2 + (3C - B)x^3 + (3D - C)x^4 \text{ \&c.}$$

which would give $0 = -1$, $A = \frac{0}{3}$ &c. The first of these is absurd, and from the others nothing can be determined; the form of the assumed series must therefore be modified.

Putting $\frac{1}{3x - x^2}$, under the form $\frac{1}{x} \times \frac{1}{3 - x}$, let the assumed series express the latter factor; then

$$\frac{1}{x} \times \frac{1}{3 - x} = \frac{1}{x}(A + Bx + Cx^2 + Dx^3) \text{ \&c.}$$

from which we find, by the usual process,

$$A = \tfrac{1}{3},\ B = \tfrac{1}{9},\ C = \tfrac{1}{27},\ D = \tfrac{1}{81}, \text{ \&c.}$$

then

$$\frac{1}{3x - x^2} = \frac{1}{x}(\tfrac{1}{3} + \tfrac{1}{9}x + \tfrac{1}{27}x^2 + \tfrac{1}{81}x^3 \text{ \&c}).$$

or

$$\frac{1}{3x - x^2} = \tfrac{1}{3}x^{-1} + \tfrac{1}{9}x^0 + \tfrac{1}{27}x^1 + \tfrac{1}{81}x^2 \text{ \&c.}$$

Again, if we assumed

$$\frac{1 - x^2}{1 + x^2 - x^4} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6, \text{ \&c.,}$$

we should find the true series to be

$$1 - 2x^2 + 3x^4 - 5x^6 + 8x^8, \text{ \&c.,}$$

the coefficients B, D, F, &c. of the odd powers of x becoming zero. The series above assumed will, therefore, answer the purpose as well as if we had assumed

$$\frac{1 - x^2}{1 + x^2 - x^4} = A + Cx^2 + Ex^4 + Gx^6, \text{ \&c.,}$$

in which the terms that contain the odd powers are wanting.

162. The following miscellaneous examples will serve to illustrate some of the applications of this method:

Ex. 1. Required the development of $\sqrt{(a^2 + x^2)}$.

Assume $\sqrt{(a^2 + x^2)} = A + Bx + Cx^2 + Dx^3 + \text{\&c.}$

Squaring both sides so as to obtain quantities of the same form, we find

$$a^2 + x^2 = A^2 + 2ABx + \left.\begin{array}{r}2AC\\ \\ +B^2\end{array}\right| x^2 + \left.\begin{array}{r}2AD\\ \\ +2BC\end{array}\right| x^3 + \left.\begin{array}{r}2AE\\ +2BD\\ +C^2\end{array}\right| x^4 + \&c.$$

Hence, equating the corresponding coefficients, we get

$$A^2 = a^2,\ 2AB = 0,\ 2AC + B^2 = 1,\ 2AD + 2BC = 0,\ \&c.$$

The first of these gives $A = a$, the second $B = 0$; the others give by substitution the respective values, $C = \frac{1}{2a}$, $D = 0$, $E = -\frac{1}{8a^3}$ &c., the series required therefore is

$$\sqrt{(a^2 + x^2)} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3}\ \&c.$$ See Example 1, Art. 99.

2. Let $x = ay + by^2 + cy^3 + \&c.$; required the value of y in terms of x.

Assume $y = Ax + Bx^2 + Cx^3 + \&c.$

$$\left.\begin{array}{llll}\text{Then } ay = & aAx + & aBx^2 + & aCx^3 + \&c. \\ by^2 = & & bA^2x^2 + & 2bABx^3 + \&c. \\ cy^3 = & & & cA^3x^3 + \&c. \\ \&c. & & & \&c. \\ -x = -x & & & \end{array}\right\} = 0;$$

hence

$$aA - 1 = 0, \text{ or } A = \frac{1}{a};\ aB + bA^2 = 0, \text{ or } + B = -\frac{b}{a^3}$$

$$aC + 2bAB + cA^3 = 0, \text{ or } C = \frac{2b^2 - ac}{a^5}\ \&c.$$

We have therefore

$$y = \frac{x}{a} - \frac{bx^2}{a^3} + \frac{(2b^2 - ac)x^3}{a^5} + \frac{5b^3 - 5abc + a^2d}{a^7}x^4\ \&c.$$

3. Let it be required to resolve the fraction $\frac{6x^3-2x-6}{(x^2+1)(x+2)(x-3)}$ into partial fractions, whose denominators shall be the factors of the given denominator.

Assume $\frac{6x^3-2x-6}{(x^2+1)(x+2)(x-3)}=\frac{A+Bx}{x^2+1}+\frac{C}{x+2}+\frac{D}{x-3}$.

Adding the partial fractions, and comparing the coefficients of the like powers of x in the numerator of the results, and in that of the given fraction, we find the following series of equations for determining the assumed coefficients:

$$B+C+D=6,\ A-B-3C+2D=0,$$
$$A+6B-C-D=2,\ 6A+3C-2D=6.$$

These equations give $A=1$, $B=1$, $C=2$, $D=3$; and the partial fractions therefore are

$$\frac{x+1}{x^2+1},\ \frac{2}{x+2},\ \frac{3}{x-3}.$$

4. Resolve $\frac{A+Bx+Cx^2}{(1+ax)(1+bx)(1+cx)}$ into its partial fractions.

Assume $\frac{A+Bx+Cx^2}{(1+ax)(1+bx)(1+cx)}=\frac{P}{1+ax}+\frac{Q}{1+bx}+\frac{R}{1+cx}$; then

$A+Bx+Cx^2=P(1+bx)(1+cx)+Q(1+ax)(1+cx)+R(1+ax)(1+bx)$.

The equation being identical, or true for any value of x, let $1+ax=0$, or $x=-\frac{1}{a}$, then

$$A-\frac{B}{a}+\frac{C}{a^2}=P\left(1-\frac{b}{a}\right)\left(1-\frac{c}{a}\right);$$

whence $$P=\frac{Aa^2-Ba+C}{(a-b)(a-c)}.$$

Similarly if $1 + bx = 0$, or $x = -\frac{1}{b}$, and $1 + cx = 0$, or $x = -\frac{1}{c}$, we shall find

$$Q = -\frac{Ab^2 - Bb + C}{(a - b)(b - c)}; \text{ and } R = \frac{Ac^2 - Bc + C}{(a - c)(b - c)}.$$

The three partial fractions will be obtained by dividing these values of P, Q, and R, by $1 + ax$, $1 + bx$, and $1 + cx$, respectively.

5. Resolve $7x^2 - 6x - 1$ into two factors of the first degree.

Here, since -1 is the product of $+1$ and -1, we assume for the factors, $Ax + 1$, and $Bx - 1$; then the product of these, $ABx^2 - (A - B)x - 1$ must be the same as $7x^2 - bx - 1$; $\therefore$ $AB = 7$, and $A - B = 6$; $\therefore$ also $A^2 - 2AB + B^2 = 36$; adding to this $4AB = 28$, we get $A^2 + 2AB + B^2 = 64$ $\therefore$ $A + B = 8$. Hence $2A = 14$, and $A = 7$; $\therefore$ $B = 1$; and the factors are $7x + 1$ and $x - 1$. The resolution may also be effected by means of quadratics; we thus find the roots to be 1 and $-\frac{1}{7}$, and hence (Art. 157), the factors are $7x + 1$, and $x - 1$.

The preceding examples, as well as some of the exercises that follow, will serve to illustrate the nature of two important branches of analysis, the decomposition of rational fractions, and the reversion of series; it is impossible to enter further on the subject here.

EXERCISES.

Convert into series,

1. $\dfrac{a}{b + cx}$
2. $\dfrac{a + bx}{a' + b'x + c'x}$
3. $\dfrac{1}{1 - 2ax + x^2}$
4. $\sqrt{(1 + x^2)}$
5. $\sqrt{(1 - x)}$

6. $\dfrac{a}{1+2x+x^2}$

7. $\dfrac{3+2x}{5+7x}$

8. $\dfrac{1+2x}{1-x-x^2}$

9. $\dfrac{1}{1+x+x^2+x^3}$

Resolve into partial fractions,

10. $\dfrac{x+3}{(x-1)(x+2)}$

11. $\dfrac{x+1}{x^2-7x+12}$

12. $\dfrac{x^2}{(x^2-1)(x-2)}$

13. $\dfrac{1}{(x+a)(x+b)(x+c)}$

14. $\dfrac{1}{x^4-a^4}$

15. $\dfrac{1}{x^3-x^2-2x}$

Resolve into two factors of the first degree,

16. $12x^2+5x-25$.

17. $2x^2-21xy-11y^2-x+34y-3$.

Required the value of y in a series of ascending powers of x.

18. $x = y - ay^3 + by^5 - \&c.$

19. $x = y - \dfrac{y^2}{2} + \dfrac{y^3}{3} - \&c.$

20. $y^3 - 3y + x = 0$.

21. $x = 2y + 3y^3 + 4y^5 + \&c.$

22. $x = 1 + y + \dfrac{y^2}{1\,.\,2} + \dfrac{y^3}{2\,.\,3} + \dfrac{y^4}{2\,.\,3\,.\,4} + \&c.$

XIX.—RATIO AND PROPORTION.

163. *Ratio* is the relation which two quantities of the same kind have to one another, with respect to the number of times the one is contained in the other.

Ratio is expressed by placing between the quantities the common sign of division, or else two dots (:). Thus, the ratio of a to b is expressed by $a : b$, $a \div b$, or $\frac{a}{b}$.

164. A ratio is not altered by multiplying or dividing its terms by the same quantity (Art. 68).

165. The terms of a ratio are called the *antecedent* and *consequent*.

166. If the second of four quantities is contained in the first as often as the fourth in the third, the first is said to have to the second *the same ratio* as the third to the fourth.

167. When the second of four quantities is contained in the first oftener than the fourth in the third, the first is said to have to the second a *greater ratio* than the third to the fourth.

Thus, $4 : 2$ is a greater ratio than $4 : 3$, and $8 : 5$ a less ratio than $7 : 4$; because $\frac{12}{6} > \frac{8}{6}$, and $\frac{32}{20} < \frac{35}{20}$.

And generally, we may compare two or more ratios by reducing the fractions which measure them to the same denominator; that which has the greater numerator is greater than the other. Thus, suppose the ratios to be $a : b$, and $c : d$; then $\frac{a}{b} = \frac{ad}{bd}$, and $\frac{c}{d} = \frac{bc}{bd}$; and the first ratio will be greater than, equal to, or less than, the second ratio, according as ad is greater than, equal to, or less than bc.

168. A ratio is called a ratio of *greater inequality*, of *less inequality*, or of *equality*, according as the antecedent is *greater* than, *less* than, or *equal* to, the consequent.

169. *A ratio of greater inequality is diminished, and one of less inequality increased, by adding any quantity to both its terms.*

Thus, of the ratios 8 : 5, and 7 : 4, whose terms differ by a unit, 8 : 5 is the greater (Art. 167), and generally if x be added to the terms of the ratio $a : b$, it becomes $a + x : b + x$; then $\frac{a+x}{b+x}$ is greater or less than $\frac{a}{b}$, according as $b\ (a + x)$ is greater or less than $a(b + x)$; that is, according as xb is greater or less than xa; that is, according as b is greater or less than a.

170. *A ratio of greater inequality is increased, and one of less inequality is diminished, by taking from both terms any quantity less than each.*

For, as before, $\frac{a-x}{b-x}$ is greater or less than $\frac{a}{b}$, according as $b(a - x)$ is greater or less than $a(b - x)$, that is, according as bx is less or greater than ax, that is, according as b is less or greater than a.

171. If the difference between the terms of a ratio be very small compared with either of these, the ratio of their squares is nearly obtained by doubling this difference.

Let $a + x : a$ be the ratio; then in the ratio $a^2 + 2ax + x^2 : a^2$, or $a + 2x + \frac{x^2}{a} : a$; since x is very small compared to a, $\frac{x^2}{a}$ will be an extremely small fraction, when compared either with a or $2x$, and it may therefore be neglected without sensibly affecting the ratio; that is, $a + 2x : a$ expresses the ratio of the squares very nearly.

Similarly $a + 3x : a$ expresses the ratio of the third powers very nearly; $a + \frac{1}{2}x : a$, and $a + \frac{1}{3}x : a$ the ratios of the square roots and cube roots very nearly.

Thus the ratio of $1001^2 : 1000^2$ is nearly $1002 : 1000$, the real ratio being $1002\cdot001 : 1000$. Also the ratio of $(1\cdot7361)^4$ to $(1\cdot736)^4$ is that of $1\cdot7360 + 4 \times 0\cdot0001$ to $1\cdot736$ nearly, or of $1\cdot7364$ to $1\cdot736$.

172. The ratio of one quantity to another is said to be *compounded* of two or more ratios, when the number which is the measure (Art. 46) of the ratio in question is equal to the product of the measures of the other ratios.

Thus, the ratio of 3 to 108 is compounded of $6 : 3$, $8 : 4$, $15 : 5$, $12 : 4$, or of any others whose measures are factors of 36. Also, the ratio $ace : bdf$ is compounded of $a : b$, $c : d$, $e : f$.*

173. A compound ratio whose measure is the product of two equal factors, that is, the second power, of the measure of another ratio, is called the *duplicate* of that ratio. In like manner, the ratio of two quantities is said to be the *triplicate* ratio of two other quantities, when its measure is the third power of the measure of the ratio of two other quantities. Thus, $3 : 75$ is duplicate of $4 : 20$, and $3 : 192$ is triplicate of $4 : 16$.

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* For if b be contained x times in a, and d be contained y times in c, and f be contained z times in e, bdf, the product of the consequents will be contained xyz times in ace, the product of the antecedents. Hence we might define compound ratio thus: In any number of ratios, if the product of the antecedents be taken, as also that of the consequents, the products are said to have to one another a ratio compounded of the other ratios.

$a : b$
x
$c : d$
y
$e : f$
z

174. *Proportion*, or *analogy*, is the equality of ratios, and quantities having the same ratio, are called *proportionals*. Thus, if $a : b$ be equal to $c : d$, these quantities constitute a proportion, which is written $a : b :: c : d$, $a : b = c : d$, or $\frac{a}{b} = \frac{c}{d}$.

175. A *continual proportion* is a series of terms such that each term has the same constant ratio to the one next following. Thus, a, ar, ar^2, ar^3, ar^4, ar^5, ... ar^{n-1}, is a series of continual proportionals, in which r is the common ratio, and ar^{n-1} the last term.

176. If three quantities be continual proportionals, the ratio of the first to the third is duplicate of the ratio of the first to the second; and if there be four quantities, the ratio of the first to the fourth is triplicate of the ratio of the first to the second. For, let there be the two series,

$$ar^2,\ ar,\ a\,;$$
$$ar^3,\ ar^2,\ ar,\ a\,;$$

then the measure of $ar^2 : a$ is r^2, and of $ar^2 : ar$ is r; also, the measure of $ar^3 : a$ is r^3, and of $ar^3 : ar^2$ is r. Hence the ratio of the squares is duplicate, and of the cubes triplicate, of the ratio of the first powers.*

177. Three quantities are said to be in *harmonical* proportion, when the first has the same ratio to the third that the difference of the first and second has to the difference of the second and third. Thus, a, b, c, are in harmonical proportion, when $a : c :: a - b : b - c$. When every three con-

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* The ratio of the square roots is called *sub-duplicate*, and of the cube roots *sub-triplicate* of the ratio of the first powers. Also the ratio of the square roots of the third powers ($a^{\frac{3}{2}} : b^{\frac{3}{2}}$), is called sesquiplicate of the ratio of the quantities ($a : b$).

secutive terms are thus related, the series forms an harmonic progression. Thus, a, b, c, d, e, &c., form an harmonic progression, if $a : c :: a - b : b - c$, if $b : d :: b - c : c - d$, and $c : e :: c - d : d - e$, &c.

178. *If four quantities be proportionals, the product of the means is equal to the product of the extremes; and if the product of two quantities be equal to the product of two others, the four quantities will form a proportion.*

Let the proportion be $a : b :: c : d$; then $\frac{a}{b} = \frac{c}{d}$; and clearing of fractions, $ad = bc$. Again, let $mq = np$; divide both sides by nq, then $\frac{m}{n} = \frac{p}{q}$, or $m : n :: p : q$.

179. Hence, if the proportion consist of three terms, the product of the extremes is equal to the square of the mean, and conversely. Hence, also, a mean proportional between two quantities is the square root of their product; for in the proportion $a : b :: b : c$, $b^2 = ac$, and $b = \sqrt{ac}$. This is called a *geometric mean.*

180. Hence, also, any three terms in a proportion being given, we may find the fourth from the equation $ad = bc$.

181. *If four quantities be proportionals, they are also proportionals when taken alternately or inversely.*

Let $a : b :: c : d$, or $\frac{a}{b} = \frac{c}{d}$, be the proportion, and let both sides be multiplied by $\frac{b}{c}$; then $\frac{a}{c} = \frac{b}{d}$, and therefore $a : c :: b : d$. This alternation cannot take place (Art. 163) unless the quantities are of the same kind.

Again, let $a : b :: c : d$; hence (Art. 178), $ad = bc$, and therefore $b : a :: d : c$.

182. *In a proportion the first term is to the second increased or diminished by any multiple of the first, as the third is to the fourth increased or diminished by any multiple of the third.*

Let $a : b :: c : d$; then inversely (Art. 181), $b : a :: d : c$; hence, $\frac{b}{a} = \frac{d}{c}$, and (Arts. 27, 28) $\frac{b}{a} \pm m = \frac{d}{c} \pm m$:

whence $$\frac{b \pm ma}{a} = \frac{d \pm mc}{c},$$

and therefore, $b \pm ma : a :: d \pm mc : c$;
and inversely, $a : b \pm ma :: c : d \pm mc$.

183. Hence, $a : c :: b \pm ma : d \pm mc$, therefore, $\frac{a}{c} = \frac{b \pm ma}{d \pm mc}$; but $\frac{a}{c} = \frac{b}{d}$, therefore $b : d :: b \pm ma : d \pm mc$; and alternately, $b : b \pm ma :: d : d \pm mc$.

Also, since $\frac{a}{c} = \frac{b + ma}{d + mc}$, and $\frac{a}{c} = \frac{b - ma}{d - mc}$, we have alternately, $b + ma : b - ma :: d + mc : d - mc$.

Hence, if $m = 1$, $b + a : b - a :: d + c : d - c$; and from the last proportion in Art. 182, $a : b \pm a :: c : d \pm c$, and $b \pm a : a :: d \pm c : c$. Also, $b : b \pm a :: d : d \pm c$, and $b \pm a : b :: d \pm c : d$.

184. *When any number of quantities are proportionals, as one antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.*

Let there be any number of proportionals,
$$a : b :: c : d :: e : f;$$
then $ad = bc$, and $af = be$; also $ab = ba$; therefore,
$$ab + ad + af = ba + bc + be,$$
or $$a(b + d + f) = b(a + c + e);$$
whence (Art. 178), $a : b :: a + c + e : b + d + f$.

185. *If the first and second terms of a proportion be multiplied or divided by any quantity, as also the third and fourth, the quantities which result will be proportional.*

Let the proportion be $\frac{a}{b}=\frac{c}{d}$; therefore (Art. 68),

$$\frac{ma}{mb}=\frac{\frac{c}{n}}{\frac{d}{n}}; \qquad \text{whence, } ma:mb::\frac{c}{n}:\frac{d}{n}.$$

So also if $a:b::c:d$;

then $$a:b::\frac{1}{d}:\frac{1}{c}.$$

For, as above shewn,

$$a:b::\frac{c}{cd}:\frac{d}{cd},$$

that is $$a:b::\frac{1}{d}:\frac{1}{c}.$$

So that in a proportion, the first is to the second as the reciprocal of the fourth to the reciprocal of the third. We have also $a:c::b:d$, and $\therefore a:c::\frac{1}{d}:\frac{1}{b}$. *See* Euclid's definition of "reciprocally proportional," VI., Def. 2.

186. *If the first and third terms of a proportion be multiplied or divided by any quantity, as also the second and fourth, the quantities which result will be proportional.*

For let $a:b::c:d$, then $\frac{a}{b}=\frac{c}{d}$, and multiplying both sides of this equation by $\frac{m}{n}$,

$$\frac{ma}{nb}=\frac{mc}{nd} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1);$$

whence $ma : nb :: mc : nd.$[*]

Also if $n = 1$ in (1), $ma : b :: mc : d,$

and if $m = 1$, $a : nb :: c : nd,$

we have also $ma : \frac{b}{n} :: mc : \frac{d}{n}$ and $a : \frac{b}{n} :: c : \frac{d}{n}$, &c.

[*] Hence ma is greater than, equal to, or less than nb, according as mc is greater than, equal to, or less than nd. This proves Euclid's definition of "same ratio" (V. 5); and thus we deduce the geometrical definition from the algebraical. The algebraical definition (Arts. 163 and 166) could not be employed in geometry, as there is no geometrical method of representing a quotient.

But Euclid's definition is more rigorous and comprehensive than the algebraical, for it applies to *incommensurable*, as well as to *commensurable* quantities; while the algebraical (Arts. 163, 166) applies to the latter only. For if the quantities compared be incommensurable, such as the circumference and diameter of a circle, the diagonal and side of a square, &c.; or if we have such a ratio as $3 : \sqrt{2}$, that is, $3 : 1·4142$ . .; the measure of the ratio cannot be expressed, either by a whole number or by any determinate fraction. In such a case, however, a fraction may be found which shall express the value of the ratio $a : b$ with any required degree of accuracy. For let x be a measure of b, so that $b = nx$, where n is an integer. Also, let a be greater than mx but less, than $(m + 1)\,x$; then $\frac{a}{b}$ is greater than $\frac{m}{n}$, but less than $\frac{m+1}{n}$; or the difference between $\frac{a}{b}$ and $\frac{m}{n}$ is less than $\frac{1}{n}$; and since $nx = b$, as x is diminished, n is increased, and $\frac{1}{n}$ diminished. Therefore by continually diminishing x more and more, $\frac{1}{n}$ may be continually diminished; so that by taking x small enough, $\frac{1}{n}$ and also the difference between $\frac{m}{n}$ and $\frac{a}{b}$ may be made less than any assignable magnitude; and, therefore, a fraction may be found expressing the measure of the ratio of a to b, *to any required degree of accuracy.*

Again, if c and d also are incommensurable, and if when $\frac{a}{b}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, $\frac{c}{d}$ also lies between the same limits; then, however m and n are increased, $\frac{a}{b}$ is equal to $\frac{c}{d}$.

187. *In any number of proportions, if the corresponding terms be multiplied together, the products will be proportional.*

Let there be any number of proportions,

$$a : b :: c : d,$$
$$e : f :: g : h,$$
$$i : k :: l : m,$$

then $$\frac{a}{b}=\frac{c}{d}, \frac{e}{f}=\frac{g}{h}, \frac{i}{k}=\frac{l}{m};$$

hence, $$\frac{aei}{bfk}=\frac{cgl}{dhm};$$

and therefore $$aei : bfk :: cgl : dhm.$$

This is called *compounding* the proportions (Art. 172).

188. *If four quantities be proportionals, their like powers and roots will be proportionals.*

Let $a : b :: c : d$, ∴ (Arts. 31, 68), $\frac{a}{b}=\frac{c}{d}$ and $\frac{a^n}{b^n}=\frac{c^n}{d^n}$; where n may be either a whole number or a fraction.

Hence (Art. 174), $$a^n : b^n :: c^n : d^n.$$

For if they are not equal, their difference must be some assignable quantity less than $\frac{1}{n}$, because each of them lies between $\frac{m+1}{n}$ and $\frac{m}{n}$, whose difference is $\frac{1}{n}$. Now n may be increased without limit, according to the supposition, and therefore $\frac{1}{n}$ diminished without limit, that is, made *less* than any assignable quantity, and $\frac{a}{b}$ and $\frac{c}{d}$ thus have no assignable difference; that is, we may say that $\frac{a}{b}$ is equal to $\frac{c}{d}$; so that a, b, c, d form a proportion in this case as well as in that supposed in our definition; and hence all the propositions respecting proportionals are true in regard to these four quantities.

XX. VARIATION.

189. One quantity is said to *vary directly* as another, when, as the one increases or diminishes, the other increases or diminishes in *the same proportion.*

Thus, if the altitude of a triangle be invariable or constant (Euc. VI. 1), the area will vary as the base; and (Euc. VI. 33) in equal circles, or in the same circle, angles, whether at the centre or circumference, vary as the arcs on which they stand. But one quantity may change with another, without varying in the same proportion. Thus, in the expression $y = \sqrt{ax}$, y increases or diminishes with x; but is not doubled or trebled when x is doubled or trebled; y does not increase *in the same proportion* as x, and so is not said to *vary as* x.

The sign $\propto$ (read *varies as*) is employed to denote the relation of varying and dependent quantities. Thus, $y \propto x$ means that y varies as x.

The sign $\doteqdot$ (read *proportional to*) would be a very suitable symbol to express this variation, as it includes the signs of equality and ratio, denoting an equality of ratios.

190. One quantity is said to *vary inversely* as another, when they are so related that the one changes *in the same proportion* as the *reciprocal* of the other (Art. 79).

Thus, if the area of a triangle be given, the base *varies inversely* as the altitude, that is, its *reciprocal* varies as the altitude. For if we call p and p' the altitudes of two equal triangles, and b and b' their bases, $\frac{1}{2}(bp) = \frac{1}{2}(b'p')$. Hence, $bp = b'p'$, and therefore (Art. 178),

$$b : b' :: p' : p$$

or (Art. 185),

$$b : b' :: \frac{1}{p} : \frac{1}{p'}$$

that is, $b \propto \frac{1}{p}$, or $p \propto \frac{1}{b}$.

Generally, this variation is expressed thus, $y \propto \frac{1}{x}$[*]

From the definition here given, and from that of Art. 189, it follows that when y varies inversely as x, it varies directly as $\frac{1}{x}$ Now $\frac{1}{x}$ is increased when x is diminished, and diminished when x is increased; so that when y varies inversely as x, or directly as $\frac{1}{x}$, y must become larger or smaller, according as x becomes smaller or larger.

Questions in simple and compound proportion in arithmetic, or the single and double rule of three, afford examples of direct and inverse variation. In fact, variation is nothing else than an abridged mode of presenting these subjects; the phraseology is different, and more suitable in many applications of the doctrine of proportion, while greater conciseness is introduced by expressing only two terms in each proportion; the student must bear in mind, however, that four are always supposed. Thus, the general expression

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[*] Thus also, the interest which a person will obtain for his money by buying into the government stocks, in which the dividends (3, 3½, &c., per cent) are invariable or constant, will vary inversely as the price of the stock; it will be 3 per cent (in the 3 per cents) if the stock is at £100, 4 per cent if the stock is at £75, 5 per cent if the stock is at £60, and so on. And in the case of two uniform motions, in which s, t, and v, s', t', and v', are the spaces, times, and velocities, respectively; since $v = \frac{s}{t}$ and $v' = \frac{s'}{t'}$; if the spaces are the same in both, v varies as the reciprocal of t. Thus, if two steamers run a distance of 120 miles in 6 hours and 8 hours, respectively; the velocities are 20 miles per hour, and 15 miles per hour; and we have the proportion,

$$20 : 15 = \frac{1}{6} : \frac{1}{8}.$$

that is, the velocities vary as the reciprocals of the times.

$y \propto x$ denotes that particular values of these quantities are proportionals; that is, it denotes any proportion such as

$$y' : y'' = x' : x'';$$

and the expression $$y \propto \frac{1}{x}$$

any analogy such as

$$y' : y'' = \frac{1}{y'} : \frac{1}{y''}.$$

In the subjects referred to, there are given corresponding values of two quantities which *vary with one another*, and we are required to find the change produced in one, when a given change is made in the other. Thus in the question, "If 112 cwts. cost £63, what will 192 cwts. cost at the same rate;" corresponding quantities in weight and price are given, and we have to find what change will be produced on the price when a *given* change is made on the weight. The relation is expressed by an analogy of this form,

$$w' : w'' :: p' : p'';$$

or more simply under the form,

$$w \propto p.$$

Again, in the question, "How long will 18 horses take to plough a farm which 12 horses can plough in 15 days." The time *varies* as the number of horses *inversely*; so that instead of 12 and 15 being the antecedents of the ratios, these are $\frac{1}{12}$ and 15; $\frac{1}{18}$ is the first consequent, and therefore 10 the second; for we have the analogy $\frac{1}{12} : \frac{1}{18} :: 15 : x$, or, $18 : 12 :: 15 : x, \therefore x = 10$. We have, therefore, the general expression, $$t \propto \frac{1}{\text{H}}, \text{ or, } \text{H} \propto \frac{1}{t}.$$

191. Let x and y be two variable quantities, so related

that $y \propto x$, then will $y = mx$, where m is invariable or constant.

For, since y changes in the same proportion as x changes, the ratios $y : x$, $y' : x'$, &c., in every system of values will be the same; so that the quantity $\frac{y}{x}$, which is the measure of this ratio, will be always of the same value, or invariable; it may, therefore, be denoted by a constant, m; so that

$$y = mx.$$

Thus, if ϕ, b, p, be the area, base, and altitude of a triangle, $\phi = \frac{1}{2} . bp$; and if we take as constant the altitude $= 2m$; then $\phi = m \times$ base. Also, if we take as constant the base $= 2m$, then $\phi = m \times$ altitude.

192. It follows from these considerations, that if we know any system of values of x and y, the constant m can be found.

Thus, if we know that when $x = a$, $y = b$; then $b = ma$, $\therefore\ m = \frac{b}{a}$, and $y = \frac{b}{a}x$. This is called *the equation connecting* y and x.

193. Again, conversely, if $y = mx$, we have in every system of values $\frac{y}{x} = m$, and therefore when x changes, y must also change in the same proportion; hence, $y \propto x$.

194. One quantity is said to *vary as two others jointly*, when, if the former is changed in any manner, the *product* of the other two is changed *in the same proportion*.

Thus, the area of a triangle varies as the base and perpendicular jointly; that is, the area changes in the same proportion as the product of these two quantities. If ϕ, b, p, and φ', b', p', represent these three quantities in two triangles,

then $$\varphi :: \tfrac{1}{2}.bp, \text{ and } \varphi' = \tfrac{1}{2}.b'p' \therefore \frac{\varphi}{\varphi'} = \frac{bp}{b'p'};$$

hence $$\varphi : \varphi' :: bp : b'p'; \text{ or, } \quad \varphi \propto bp.$$

195. If $z \propto xy$, it will follow, as in Art. 191, that $z = mxy$.

And conversely, it might be shewn, as in Art. 193, that if $z = mxy$, then $z \propto xy$.

196. If x, y, z, are three variables so related that a change in the product of x and $\frac{1}{y}$ changes z in the same proportion, then z is said to *vary directly* as x, and *inversely* as y. The expression $z \propto \frac{x}{y}$ represents this relation, and z plainly varies jointly as x and $\frac{1}{y}$.

Thus, the base of a triangle varies as the area directly, and the altitude inversely.

For, since $\frac{\varphi}{\varphi'} = \frac{bp}{b'p'}$ if we multiply both sides by $\frac{p'}{p}$ we

shall have $$\frac{\varphi p'}{\varphi' p} = \frac{b}{b'} = \frac{\varphi}{p} \div \frac{\varphi'}{p'};$$

and therefore, $$b : b' :: \frac{\varphi}{p} : \frac{\varphi'}{p'},$$

that is, b varies as φ directly, and p inversely.

Thus, also, in the question, "If 12 masons in 5 days can build a wall 16 poles in length; how many masons would be required to build a similar wall 24 poles long in 9 days;" we have a single term, masons, which varies separately as *each* of two others; *directly* as the length of the wall, and *inversely* as the number of days; or $\text{M} \propto \frac{\text{P}}{\text{D}}$.

We have then $\frac{16}{5} : \frac{24}{9} :: 12 : 10,$

or, $16 \times 9 : 24 \times 5 :: 12 : 10.$

197. If one quantity varies as a second, and that second as a third, the first will vary as the third.

For let $y = mx$, and $x = nz$; then $y = mnz$, and m and n being supposed constants, mn is constant, $\therefore\ y \propto z$.

Also, if $y \propto x$, and $x \propto \frac{1}{z}$; then $y \propto \frac{1}{z}$. For, let $y = mx$, and $x = \frac{n}{z}$, $\therefore\ y = \frac{mn}{z}$, and mn being constant, $y \propto \frac{1}{z}$.

198. If each of two quantities vary as a third, their sum or difference, or the square root of their product, will vary as the third.

For, let $y = mz$, and $x = nz$, m and n being constant, then $y \pm x = (m \pm n)z$, $\therefore\ y \pm x \propto z$.

Also, $\sqrt{yx} = \sqrt{(mnz^2)} = z\sqrt{mn}$, $\therefore\ \sqrt{yx} \propto z$.

199. If y varies as x, and z as v, then will yz vary as xv.

For, let $y = mx$, and $z = nv$ $\therefore\ yz = mnxv$ $\therefore\ yz \propto xv$.

Similarly, $\frac{y}{z} \propto \frac{x}{v}$.

Also, if y vary as xz, then will x vary as $\frac{y}{z}$, and z as $\frac{y}{x}$.

For, let $y = mxz$, m being constant;

then, $x = \frac{1}{m} \cdot \frac{y}{z}$, and $z = \frac{1}{m} \cdot \frac{y}{x}$,

hence, $x \propto \frac{y}{z}$, and $z \propto \frac{y}{x}$.

200. Like powers or roots of quantities vary as those quantities.

For, let $y = mx$, then $y^n = m^n x^n$, $\therefore\ y^n \propto x^n$.

201. If x, y, z, are three variables, and if z varies as x when y is supposed constant, and if z varies as y when x is supposed constant; then z varies as xy when x and y both vary.

The variation of z depends upon the variations of y and x, and if we suppose the variations to take place separately, and whilst y is changed to y', let z be changed to z''; then by hypothesis

$$z : z'' = y : y';$$

but this value, z'', will again be changed to z' by the variation of x, and proportionally to the change in x, so that

$$z'' : z' = x : x'.$$

Compounding these two proportions (Art. 187), we have

$$zz'' : z'z'' = yx : y'x',$$

or by reduction,

$$z : z' = yx : y'x',$$

that is, z varies as yx.

Thus the area of a rectangle will vary as the base, when the perpendicular is constant; and as the perpendicular when the base is constant; but as their product, if both vary.*

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* Let $\varphi = bp$; the relations will be as follows—

$$\begin{array}{rrcrcl} p & & b & & \varphi & \qquad\text{and} \\ 6 & \times & 8 & = & 48 & \quad 8 : 9 :: 48 : 54 \\ 6 & \times & 9 & = & 54 & \quad : 11 :: : 66 \\ 6 & \times & 11 & = & 66 & \quad 13 :: : 78 \\ 6 & \times & 13 & = & 78 & \quad 17 :: : 102 \\ & & \&c. & & & \qquad \&c. \end{array}$$

Our former illustration affords another example. "If 12 masons in 5 days build a wall 16 poles in length; how many poles' length of a similar wall will 10 masons build in 9 days?" If the times were the same, the length would vary with the number of masons; and if the number of these latter were the same, the work done would vary with the time; when both the time and number of men are different, the work will vary as their product; that is, the number of poles required : 16 poles $:: 9 \times 10 : 5 \times 12$; or,

$$\left.\begin{matrix} 5 : 9 \\ 12 : 10 \end{matrix}\right\} :: 16 : x; \quad \therefore x = 24.$$

In the same manner, if there be any number of quantities, x, y, z, t, &c., each of which varies as another, v, when the rest are constant; v will vary as their product when they are all changed. For we shall manifestly have

$$v : v' :: x : x'; \; v' : v'' :: y : y'$$
$$v'' : v''' :: z : z'; \; v_3 : v_4 :: t : t'$$
$$\therefore v : v_4 :: xyzt : x'y'z't'; \text{ that is, } v \propto xyzt.$$

EXERCISES IN RATIO, PROPORTION, AND VARIATION.

1. Which is the greater ratio, $3 : 5$, or $5 : 8$?

2. Which is the greater ratio, $9 : 8$, or $13 : 11$?

3. Add 1 to both terms of the ratio, $7 : 6$, and shew how the ratio is thereby affected.

4. From both terms of the ratio, $10 : 7$, subtract 1, and shew how the ratio is thereby affected.

5. Which is greater, the ratio $a^2 + b^2 : a^2 - b^2$, or $a + b : a - b$?

6. Express the ratio $\frac{1}{a} : \frac{1}{b}$, in its simplest form.

7. Express the ratio $\frac{3}{7} \cdot \frac{2}{x} : \frac{5}{9} \cdot \frac{3}{y}$ in its simplest form.

8. What ratio is sub-triplicate of that of $27 : 8$?

9. Compound the sub-duplicate ratio of $a^2 : x^2$, with the duplicate ratio of $\sqrt{a} : \sqrt{x}$.

10. Compound the sesquiplicate ratio of $a : x$, with the sub-duplicate ratio of $a : x$.

11. The ratios, $12 : 11$, $14 : 13$, $16 : 15$, express the ratio of the equatorial to the polar axis in the planets, Saturn, Jupiter, and Mars, respectively. Compare the magnitudes of these ratios.

12. Which is greater, the ratio of $a + 2 : \frac{1}{2}a + 4$, or that of $a + 4 : \frac{1}{2}a + 5$?

13. Compare the measure of the ratio compounded of the ratios 18 : 6, 24 : 4, and 12 : 3, with the measures of the simple ratios, and shew how they are related.

14. If 8, x, and 162, are three continual proportionals, find x.

15. Approximate to the ratio of $\sqrt[3]{729} : \sqrt[3]{728}$.

16. Express the ratio of $a : x$ by the reciprocals of these quantities.

17. If a have to x the duplicate ratio of $a + y$ to $x + y$, prove that y is a mean proportional between a and x.

18. If 20, x, and 45 are in continued proportion, find x.

19. Shew that 12, 16, and 24, are three, and 2, $2\frac{2}{5}$, 3, and 4, four, harmonic proportionals.

20. Compound together the duplicate ratio of 4 : 3, the triplicate ratio of 5 : 4, the sesquiplicate ratio of 16 : 4, and the sub-duplicate ratio of 441 : 225.

21. Approximate to the ratio of $501^{\frac{1}{2}} : 500^{\frac{1}{2}}$, and of $501^{\frac{1}{3}} : 500^{\frac{1}{3}}$.

22. If the ratios $a + x : b$, $a - x : c$, and $c : \dfrac{a^2 - x^2}{b}$ be compounded; and also the ratios $3x + 2 : 6x + 1$, and $2x + 3 : x + 2$; whether are the resulting ratios ratios of equality, greater inequality, or less inequality?

23. Let $y \propto x$; required the equation connecting x and y under the following conditions; when $x = 4$, let $y = 24$; when $x = 6$, let $y = 30$; when $x = a - b$, let $y = a^2 - b^2$; when $y = 15$, let $x = 3$.

24. Let $y^2 \propto a^2 - x^2$, and when $x = \sqrt{(a^2 - b^2)}$, $y = \dfrac{b^2}{a}$, find the equation connecting x and y; or what the constant m is in this case.

25. Given that $y \propto x$, and that $y = 6$, when $x = 4$; what will y be when $x = 8$?

26. Given that y is equal the sum of two quantities, whereof one varies as x, and the other inversely as x^2; and when $x = 1$, $y = 10$; and when $x = 2$, $y = 13$; express y in terms of x.

27. In the case of falling bodies, the spaces passed over are as the squares of the times, or $s \propto t^2$; when then a body has moved for 4 seconds, and described $257\frac{1}{3}$ ft., find the equation connecting the space and time.

28. Given that y varies inversely as the mth power of x, and x inversely as the nth power of z; find the equation between x and z, supposing that when $x = a$, $z = c$.

29. Suppose y to be equal to the sum of three quantities, whereof one is invariable, the other varies as x, and the third as x^2; express y in terms of x.

30. If whilst $r \propto x$, and $s \propto \sqrt{x}$, $y = r + s$; and if when $x = 4$, $y = 5$, and when $x = 9$, $y = 10$; find the equation connecting y and x.

31. If $a + b \propto a - b$, shew that $a^2 + b^2 \propto ab$.

32. Given that the volume of a sphere varies as the cube of its radius; prove that the volume of a sphere, whose radius is 6 inches, is equal to the sum of the volumes of three spheres, whose radii are 3, 4, and 5 inches.

XXI.—PROGRESSION.

202. An *equidifferent series*, called also an *arithmetical progression* is a series of terms which increase or decrease by the same *common difference*.

Thus the numbers 20, 17, 14, 11, 8, 5, 2, form an equidifferent series, and 3 is the common difference of the terms. Also a, $a + d$, $a + 2d$, $a + 3d$, $a + 4d$, &c. is another such series, where d is the common difference. It is obvious that the *n*th or last term is $a \pm (n-1)d$, according as the series is an increasing or a decreasing one. This *n*th term is also called the *general* term, because any term can be derived from it.

203. In an equidifferent series the sum of the two extremes is equal to the sum of any two terms equally distant from them, or to twice the middle term, if the number of terms be odd; for in the first of the above series we have $20 + 2 = 17 + 5 = 14 + 8 = 11 \times 2$; and in the second, if we take the first and ninth terms as extremes, we have $a + (a + 8d) = a + d + (a + 7d) = 2(a + 4d)$; the fifth being in this case the middle term.

When we have four equidifferent quantities a, b, c, d, then $a - b = c - d$; whence $a + d = b + c$; and if a, b, c be three such quantities, then $a - b = b - c$, whence $b = \frac{a+c}{2}$; and in the series already given, a, $a + d$, $a + 2d$; $a + d = \frac{a + (a + 2d)}{2}$; that is, the middle term is half the sum of the extremes. This is called an *arithmetical mean*.

204. A series of continual proportionals (Art. 175) is called a *geometrical progression;* thus, 4, 12, 36, 108, &c. is such a series, and 3 the common ratio. The general form of such series has been already given (Art. 175).

205. A mean proportional between two quantities, that is, the square root of their product, is usually called a *geometric mean* (Art. 179).*

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* The terms *arithmetical* and *geometrical*, commonly applied to proportions and progressions, are improper, as they lead to a misconception of the nature and connection of these series. The terms progressions *by differences* and *by quotients*, or *equidifferences* and *continual proportionals*, would be much more appropriate as expressive of their character.

206. To find the sum of an equidifferent series; *Multiply the sum of the extremes by the number of terms, and take half the product.*

Let a be the first term, d the common difference, and s the sum of the series; then

$$s = a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) + (a + 5d) + (a + 6d), + \&c. \;\ldots\ldots + a + (n - 1)d \;\ldots \quad (1)$$

Now the last or nth term is $a + (n - 1)d$, the last but one is $a + (n - 2)d$, the last but two $a + (n - 3)d$, the first being $a + (n - n)d$. Hence, by reversing the terms, the series may be thus written:

$$s = [a + (n - 1)d] + [a + (n - 2)d] + [a + (n - 3)d] + \ldots \&c., + a \;\ldots\ldots \quad (2)$$

Adding series (1) and series (2) together, we find

$$2s = [2a + (n - 1)d] + [2a + (n - 1)d] + [2a + (n - 1)d] + \&c. \text{ to } n \text{ terms.}$$

Whence, as each term is $2a + (n - 1)d$, and as there are n terms, we have $2s = n[2a + (n - 1)d]$,

and therefore $$s = \frac{n}{2}[2a + (n - 1)d] \;\ldots \quad (3)$$

Now let the last term $l = a + (n - 1)d \;\ldots \quad (4)$

then $$s = \frac{n}{2}(a + l) \;\ldots\ldots \quad (5)$$

If the series is a decreasing one, the common difference d is negative, the last term is $a - (n - 1)d$, and the sum

$$s = \frac{n}{2}[2a - (n - 1)d] \;\ldots \quad (6)$$

The same may be shewn more briefly thus. Let $s = a + (a + d) + (a + 2d) + (a + 3d) + \&c. \ldots\ldots + l$.

Then, reversing the terms, we have

$s = l + (l - d) + (l - 2d) + (l - 3d) + \&c. \ldots\ldots + a$,

$\therefore$ by addition $2s = (a + l) + (a + l) + (a + l) \ldots$ to n terms

$= n(a + l)$. Whence $s = \frac{n}{2}(a + l)$. Now, $l = a + (n - 1)d$, therefore $s = \frac{n}{2}[2a + (n - 1)d]$.

207. Hence, and from Art. 203, it follows that if m be the middle term of a series, $s = nm$, (7). Also, if p and q be terms equally distant from the extremes, $s = \frac{n}{2}(p + q)$ (8). Or the same may be shewn otherwise, thus: the rth term from the beginning (p) is $a + (r - 1)d$; and the rth from the end (q) is $l - (r - 1)d$, the sum of which is $a + l$: so that, $p + q = a + l$.

208. From the equations $s = \frac{n}{2}(a + l)$, and $l = a + (n - 1)d$, any two of the quantities, s, n, a, l, d, may be found, when the other three are given.

Thus, $d = \frac{l - a}{n - 1}$ (9); and also $d = \frac{2(s - an)}{n(n - 1)}$ (10). The expression for n is obtained by solving a quadratic equation. The formula for s (3) gives $2s = 2an + n^2d - nd$, whence $dn^2 + (2a - d)n = 2s$.

The solution of this equation for n gives

$$n = \frac{d - 2a \pm \sqrt{8sd + (2a - d)^2}}{2d}. \quad . \quad . \quad (11)$$

EXAMPLES.

1. Find the sum of 12 terms of the series

3, 7, 11, 15, &c.

Here, in (3) and (5) $a = 3$, $d = 4$, $n = 12$, $l = 3 + (12 - 1)4 = 47$;

hence, $s = \frac{12}{2}(3 + 47) = 50 \times 6 = 300$,

or $s = \frac{12}{2}(6 + 11 \times 4) = 300$.

2. If the first term of a progression be 14, and 28 the sum of 8 terms, what is the common difference?

Here, from (3) $28 = \frac{8}{2}(28 + 7d)$,
we find $d = -3$;
$\therefore$ the progression is 14, 11, 8, 5, 2, -1, -4, -7, &c.

If we had substituted in formula (10) we should have had $d = \frac{56 - 224}{8 \times 7} = \frac{-168}{56} = -3$ as before.

3. If the sum of an increasing series be 147, the first term 7, and common difference 7; required the number of terms.

Substituting these numbers in formula (3) we find, after some reductions, $n^2 + n = 42$; a quadratic, which gives $n = 6$, the number of terms required.

The same result may be obtained, though less readily, by substituting the given numbers in formula (11).

This gives $n = \frac{7 - 14 \pm \sqrt{(56 \times 147 + 7^2)}}{14} = -\frac{1}{2} \pm \frac{91}{14}$;
that is, $n = 6$, or -7.

The negative value here denotes that if we reckon one term more in the opposite direction, the sum will be also 147. Now the series being $7 + 14 + 21 + 28 + 35 + 42$, if we reckon another term *before* 7, we shall have

$0 + 7 + 14 +$ &c., the sum of which is also 147.

So also in the series of Ex. 79, in which the number of terms is either 7 or 13, if we stop at 7 terms the sum is 91, and if we go on to 13 terms, it is also 91; and the same explanation may be always given in the case of double integral values.

209. To find the sum of a series of continual proportionals; *Multiply the last term by the ratio, and divide the difference*

between this product and the first term by the difference between the ratio and unity; observing that the last term becomes zero, in an infinite decreasing series.

Let a be the first term, r the ratio, n the number of terms, and s the sum of the series; then

$$s = a + ar + ar^2 + ar^3 + ar^4 \;.\;.\;.\;. + ar^{n-1}.$$

Multiplying both sides by r, and from the result subtracting this equation, we have $sr - s = ar^n - a$;

whence, $$s = \frac{ar^n - a}{r - 1}: \quad .\;.\;.\;.\;.\;. \quad (1)$$

putting $ar^{n-1} = l$, this becomes $$s = \frac{rl - a}{r - 1}. \quad .\;.\;. \quad (2)$$

When r is a fraction, as n increases, r^n and ar^n will decrease; and when n becomes indefinitely great, ar^n will become less, with respect to a, than any assignable quantity, and is therefore to be taken as nothing: in this case, therefore, we have $$s = \frac{-a}{r - 1} = \frac{a}{1 - r}. \quad .\;.\;.\;.\;.\;. \quad (3)$$

which is the expression for the sum of an infinite geometric series. When we speak, then, of the sum of an infinite series of this kind, we are understood to mean that the sum of n terms of the progression can be made to differ from $\frac{a}{1 - r}$ by as small a quantity as we please; and r is in all cases supposed to be a fraction. See, in reference to this subject, the remarks in Art. 82, and the series given in that and the preceding Art.

210. From the equation $s = \frac{rl - a}{r - 1}$, any three of the quantities s, r, l, a, being given, the fourth may be found. See "*Application of Logarithms,*" Ex. 6.

211. To insert n geometric means between the extremes a and l, we have $l = ar^{n+1}$, and therefore $r^{n+1} = \frac{l}{a}$;

whence $$r = \sqrt[n+1]{\frac{l}{a}} = \left(\frac{l}{a}\right)^{\frac{1}{n+1}}.$$

EXAMPLES.

1. Find the sum of 8 terms of the series 2, 6, 18, 54, &c.

Here $a = 2$, $r = 3$, $n = 8$;

we have, therefore, by substitution in formula (1)

$$s = \frac{ar^n - a}{r - 1} = \frac{2 \times 3^8 - 2}{3 - 1} = 6560.$$

2. Find the sum of the infinite series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8}$, &c.

Here $a = 1$, $r = -\frac{1}{2}$,

hence, formula (3) $$s = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}.$$

3. Insert three means between the extremes 4 and 324.

Let the terms be a, ar, ar^2, ar^3, ar^4;

then $a = 4$, and $ar^{n+1} = ar^4 = 324$; $\therefore r^4 = 81$, and $r = 3$: hence the intermediate terms are 12, 36, 108.

4. Find the value of the recurring decimal $\cdot 245\dot{7}\dot{5}7\ldots$

Here, the terms after $\frac{24}{100}$, constitute an infinite series, whose first term is $\frac{57}{10^4}$, and common ratio $\frac{1}{10^2}$. Now, the sum of this series is $\frac{57}{9900}$; adding $\frac{24}{100}$ to this, we find, after reduction, $\frac{811}{3300}$ for the value of the given decimal.

212. To find an harmonical mean, and an harmonical third proportional.

Let a, b, c, be three harmonical proportionals; then (Art. 177), $a : c :: a - b : b - c$;

hence (Art. 178), $ab + bc = 2ac$, and $b = \dfrac{2ac}{a+c}$: . . . . (1)

that is, *an harmonical mean between two quantities is twice their product divided by their sum.*

The same equation gives, $c = \dfrac{ab}{2a - b}$, ... (2)

which is a third harmonical proportional.

213. *The reciprocals of quantities in harmonical progression are in arithmetical progression.*

Let a, b, c, d, &c., be in harmonical progression, then $a : c :: a - b : b - c$; therefore (Art. 178) $ab - ac = ac - bc$. Divide both members by abc; then $\dfrac{1}{c} - \dfrac{1}{b} = \dfrac{1}{b} - \dfrac{1}{a}$.

Again $b : d :: b - c : c - d$; therefore, $bc - bd = db - dc$. Dividing by bcd, we have $\dfrac{1}{d} - \dfrac{1}{c} = \dfrac{1}{c} - \dfrac{1}{b}$; now $\dfrac{1}{c} - \dfrac{1}{b}$ has been shewn equal to $\dfrac{1}{b} - \dfrac{1}{a}$. Thus the quantities $\dfrac{1}{a}$, $\dfrac{1}{b}$, $\dfrac{1}{c}$, $\dfrac{1}{d}$ which are the reciprocals of a, b, c, and d, have a common difference, and are therefore in arithmetical progression.

214. Since the sum of any number of quantities is not deducible from the sum of their reciprocals, there can be no general expression for the sum of an harmonic series.

215. Hence we can find the nth term of an harmonic series. For if a and b be the first two terms of such a series, $\dfrac{1}{a}$ and $\dfrac{1}{b}$ are the contiguous terms of an arithmetic progression,

and therefore $\frac{1}{b}-\frac{1}{a}$ is the common difference; hence (Art. 206), the *n*th term is

$$\frac{1}{a}+(n-1)\left(\frac{1}{b}-\frac{1}{a}\right)=\frac{(n-1)a-(n-2)b}{ab}.$$

The *n*th term of the harmonic series therefore is

$$\frac{ab}{(n-1)a-(n-2)b}$$

216. Hence also if the extremes, a and l, and n, the number of terms, be given, the intervening terms can be found; for, since the reciprocals are in arithmetic progression, $\frac{1}{l}$ will be the last term, and b being the common difference,

we shall have $$\frac{1}{l}=\frac{1}{a}+(n-1)b,$$

hence $$b=\frac{a-l}{(n-1)al}.$$

The arithmetic means are therefore,

$$\frac{1}{a}+\frac{a-l}{(n-1)al},\ \frac{1}{a}+\frac{2(a-l)}{(n-1)al}\ \&c.$$

adding these fractions, and then taking the reciprocals, we obtain for the harmonic means,

$$\frac{(n-1)al}{a+(n-2)l},\ \frac{(n-1)al}{2a+(n-3)l},\ \frac{(n-1)al}{3a+(n-4)l}\ \&c.$$

EXERCISES.

Sum the Series

1. 3, 8, 13, &c. to 18 terms.
2. 3, 7, 11, &c. to 24 terms.
3. 2, 5, 8, &c. to 17 terms.
4. 1, 2, 3, 4, &c. to 10, 18, and 24 terms.
5. 1, 3, 5, 7, &c. to 10, 18, and 24 terms.
6. 2, 4, 6, 8, &c. to 10, 18, and 24 terms.

7. 11, 9, 7, &c. to 9 terms.
8. 60, 54, 48, &c. to 10 and to 21 terms.
9. 333, 331, 329, &c. to 150 terms.
10. 50, 49, 48, &c. to 101 terms.
11. Of the natural numbers to *n* terms.
12. Of the odd numbers to *n* terms.
13. Of the even numbers to *n* terms.
14. 9, 11, 13, &c. to *n* terms.
15. 3, 6, 12, &c. to 10 and to 16 terms.
16. 1, 2, 4, 8, 16, &c. to 9 and to 14 terms.
17. 5, 20, 80, &c. to 9 terms.
18. 4, 12, 36, &c. to 10 terms.
19. 1, 3, 9, 27, &c. to 12 terms.
20. 1, -3, 9, -27, &c. to 12 terms.
21. $\frac{2}{3}$, $\frac{7}{15}$, $\frac{4}{15}$, &c. to 7 terms.
22. 1, $\frac{3}{2}$, 2, &c. to 6 terms.
23. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, &c. to 7 terms.
24. 7, $\frac{29}{4}$, $\frac{15}{2}$, &c. to 16 terms.
25. $2\frac{1}{2}$, $2\frac{5}{6}$, $3\frac{1}{6}$, &c. to 100 terms.
26. $\frac{1}{2}$, 1, $\frac{3}{2}$, &c. to 25 terms.
27. 1, 2, 3, &c., and ·1, ·2, ·3, &c., to 100 terms.
28. -1, -3, -5, &c. to 99 terms.
29. 1, $\frac{2}{3}$, $\frac{4}{9}$, &c. to 10 terms.
30. 1, $-\frac{2}{5}$, $+\frac{4}{25}$, &c. to infinity.
31. 4, 3, $\frac{9}{4}$, &c. to infinity.
32. 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, &c., and 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c., to infinity.
33. $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{9}$, &c. to infinity.
34. 1, $-\frac{2}{3}$, $\frac{4}{9}$ $-$, &c. to infinity.
35. $3\frac{3}{8}$, $2\frac{1}{4}$, $1\frac{1}{2}$, &c. to infinity.
36. $\frac{1}{\sqrt{2}}$, $\frac{1}{2}$, $\frac{1}{2\sqrt{2}}$, &c. to *n* terms; and $\sqrt{\frac{3}{2}}$, $\sqrt{\frac{2}{3}}$, $\frac{2}{3}\sqrt{\frac{2}{3}}$, &c. to infinity.
37. $\frac{1}{3}$, $(-\frac{1}{3})^3$, $(\frac{1}{3})^5$, $-$ &c. to infinity.

38. $\frac{10}{11}$, $(\frac{10}{11})^2$, $(\frac{10}{11})^3$, &c. to infinity.
39. Whose first term is 40 and ratio $\frac{3}{7}$, to infinity.
40. $\frac{1}{x}, -\frac{1}{x^2}, \frac{1}{x^3}$, – &c. to infinity (Art. 81).
41. $\frac{x}{a}, -\frac{bx}{a^2}, \frac{b^2x}{a^3}$, – &c. to infinity (Art. 81).
42. 12, $12\frac{2}{3}$, $12\frac{1}{3}$, &c. to n terms.
43. $\frac{1}{2}, -\frac{2}{3}, -\frac{11}{6}$, &c. to n terms.
44. $\frac{n-1}{n}, \frac{n-2}{n}, \frac{n-3}{n}$, &c. to n terms.
45. $(a^2 - b^2) + (a + b) + \frac{a+b}{a-b}$, &c. to n terms.
46. $\frac{n-1}{n}, \frac{n-3}{n}, \frac{n-5}{n}$, &c. to n terms.
47. In which 27 and 37 are equidistant terms, and $n = 13$.
48. In which 22 and 42 are equidistant terms, and $n = 13$.
49. In which 32 is the middle term, and $n = 13$.
50. In which 3 and 579 are extremes, and $d = 9$.

Insert

51. Four A. M. between 4 and 49.
52. Four A. M. between 193 and 443; and between 2 and – 18.
53. Three A. M. between 117 and 477; and between $\frac{1}{3}$ and $\frac{1}{2}$.
54. Three G. M. between 5 and 405; between 1·25 and 20; and between $\frac{1}{2}$ and $\frac{2}{3}$.
55. Three G. M. between 39 and 3159.
56. An H. M. between 18 and 54.
57. An H. M. between 5 and 15; and between 2 and 6.
58. An H. M. between 1 and $\frac{2}{3}$.
59. Four G. M. between $\frac{1}{9}$ and 27; and seven between 2 and 13122.
60. Three G. M. between 7 and 4375; and between 3 and 7203.

61. Two H. M. between 3 and 12; between 1 and $\frac{2}{3}$, and between 6 and 24.

62. Four H. M. between 2 and 12; and six between 3 and $\frac{6}{23}$.

63. One G. M. between 23 and 8464; between $\frac{16}{27}$ and $1\frac{1}{3}$; between $a^2 - x^2$, and $\frac{a+x}{a-x}$.

64. Seven A. M. between 1 and $-\frac{1}{2}$.

65. One A. M. between $2\frac{1}{3}$ and $4\frac{1}{12}$; and between $\frac{n-1}{n}$, and $\frac{n-3}{n}$.

Given,

66. In an A. P., $s = 72$, $a = 17$, $d = -2$; find the number of terms, and explain the double answer.

67. In an A. P., $s = 40$, $a = 7$, $d = 2$; find n, and explain the double answer; also, if $a = 3\frac{1}{3}$, $d = 1\frac{4}{9}$, and $s = 22$; and $a = 7$, $d = 2$, $s = 567$.

68. In an A. P., $171 = s$, $d = 4$, $n = 9$; find a.

69. In an A. P., $a = 2$, $s = 100$, $n = 8$; find d.

70. The first and second terms of an H. P. to be 5 and 4; find the 10th term.

71. The extremes 500 and 70, common difference 10; find the number of terms.

72. The extremes 1 and 99; number of terms 8; find the common difference.

Find

73. How many terms of the series 54, 51, 48, &c. amount to 513.

74. How many terms of the series $\frac{3}{4}$, $\frac{7}{8}$, 1, &c. amount to $60\frac{1}{8}$.

75. The common difference, the first term being 5, and the sum of 30 terms 1455.

76. An H. fourth proportional to 6, 8 and 12.

77. An H. third proportional to 3 and 2; to 6 and 9, and to 6 and 8.

78. The values of the interminate decimals $\cdot\dot{1}2\dot{3}$; $\cdot\dot{3}7\dot{5}$ and $\cdot\dot{8}\dot{6}$.

79. How many terms of the series 19, 17, 15, &c. amount to 91.

80. The 15th term, and the sum of 15 terms of the series whose first term is 8, and common ratio $\frac{1}{2}$.

81. The value of $\cdot\dot{1}4\dot{8}$; $\cdot13\dot{8}$; $\cdot56\dot{8}\dot{1}$; and $\cdot653\dot{1}$.

82. The 7th term of an A. P., the 5th and 9th being 13 and 25.

83. How many terms of the series ·034, ·0344, ·0348, &c., amount to 2·748.

84. The G. M. between x and y is to the A. M. between x and y, as $b : a$; shew that $x : y = a + \sqrt{(a^2 + b^2)} : a - \sqrt{(a^2 - b^2)}$.

85. Three quantities, a, b, c, are in A. P., G. P., or H. P., according as $\dfrac{a-b}{b-c}$ is equal to $\dfrac{a}{a}$, to $\dfrac{a}{b}$, or to $\dfrac{a}{c}$.

86. If a, b, c be in H. P., shew that $\dfrac{a}{b+c}$, $\dfrac{b}{c+a}$, $\dfrac{c}{a+b}$ are also in H. P.

87. Shew that the G. M. between two quantities is a mean proportional between the A. M., and the H. M. between the same quantities; and that the A. M. is greater than the H. M.

88. The A. M. of two numbers is 3, and the H. M. $\frac{8}{3}$; find the numbers.

89. If the A. M. between a and b, be twice as great as the G. M., shew that $a : b = 2 + \sqrt{3} : 2 - \sqrt{3}$.

90. If the A. M. between a and b be m times as great as the H. M., shew that $a : b :: \sqrt{m} + \sqrt{(m-1)} : \sqrt{m} - \sqrt{(m-1)}$.

91. Continue the two H. P. 2, 3, 6 ; $\frac{5}{17}$, $\frac{1}{2}$, $\frac{5}{8}$, to two terms both ways.

92. Required the sum of each of the following series to n terms ; $2\frac{1}{2}$, -1, $\frac{2}{5}$, &c.; and x^4, $-ax$, $\frac{a^2}{x^2}$ &c.

93. There are n arithmetic means between 1 and 31, and the 7th is to the ($n-1$th) as 5 to 9 ; find n and write out the series.

94. Required the sum of the series $\frac{\sqrt{2}+1}{\sqrt{2}-1}$, $\frac{1}{2-\sqrt{2}}$, $\frac{1}{2}$, &c., to infinity ; and of a^p, a^{p+q}, a^{p+2q} &c., to n terms.

95. If a, b, c be three quantities in H. P., shew that $a^2 + c^2 > 2b^2$.

96. The first term of an H. P. is $\frac{1}{2}$, and the sum of three consecutive terms $1\frac{1}{12}$; find the series.

97. In an A. P., if the $(p+q)$th term is m, and the $(p-q)$th is n ; find the pth and qth terms.

98. In a G. P., if the $(p+q)$th term is m, and $(p-q)$th term is n ; find the pth and qth terms.

99. Find the relation between a, b, c, that they may be the pth qth and rth terms of a G. P. ; and also the pth, qth, and rth terms of an H. P.

100. If $S_1 + S_2 + S_3 \ldots + S_p$ are the sums of p Arithmetic progressions, each continued to n terms, the first terms 1, 2, 3, &c. and the common differences 1, 3, 5, 7, &c. ; shew that $S_1 + S_2 + S_3 + \ldots\ldots + S_p = \frac{np}{2}(np+1)$.

XXII. PROBLEMS IN PROGRESSION AND VARIATION.

217. In the class of problems which follow, as in those already given in Section XVIII., the conditions of the question are first to be reduced to algebraical language; the relations among the quantities will supply the means for forming an equation embracing these conditions. The solution of the equations thus formed generally presents no difficulty; it is in the *reduction* of the question only that any difficulty will be encountered. But it is manifest from the nature of the subject, that no fixed rule can be laid down; facility in the art of reduction can only be acquired by practice, and a careful study of the conditions. With this view the questions are arranged in a series, progressively more difficult; as has already been done in the case of the former series.

EXAMPLES.

1. A mixture is made of 51 gallons, consisting of three liquids A, B, and C, the quantities of which, in the mixture, form an equidifferent series; and the number of gallons of A and B together is to the number of B and C together, as 8 to 9. Required the quantity of each liquid.

Let $x-y$, x, $x+y$, three equidifferent quantities, represent the number of gallons of each liquid; then the sum of these is $3x=51$; $\therefore x=17$; also by the question $2x-y : 2x+y = 8 : 9$, whence $y=2$; and the quantities therefore are 15, 17, 19.

2. Four numbers are in geometrical progression; the

sum of the first and second is 15, and the sum of the third and fourth is 60. Required the numbers.

Let the numbers be x, xy, xy^2, xy^3; then $x + xy = 15$, and $xy^2 + xy^3 = y^2(x + xy) = 60$. Whence $y = \pm 2$, and $x = 5$. The numbers therefore are 5, 10, 20, 40.

3. A number consists of three digits, the first of which is to the second as the second to the third; the number itself is to the sum of its digits, as 124 to 7; and if 594 be added to it, the digits will be inverted. Required the number.

The digits plainly form three continual proportionals; let them therefore be x, xy, xy^2. Then

$$100x + 10xy + xy^2 : x + xy + xy^2 :: 124 : 7;$$

by reduction,

$$100 + 10y + y^2 : 1 + y + y^2 :: 124 : 7.$$

Now in any proportion (Art. 183),

$$a - b : b :: c - d : d.$$

Hence, $\quad 99 + 9y : 1 + y + y^2 :: 117 : 7.$

This gives a quadratic by reduction, from which we find $y = 2$, or $-\frac{32}{13}$.

Also by the question $100x + 10xy + xy^2 + 594 = 100xy^2 + 10xy + x$, an equation which gives $x = 2$; so that the digits are 2, 4, 8, and the number 248.

4. If m shillings in a row reach as far as n sovereigns, and a pile of p shillings is as high as a pile of q sovereigns, compare the values of equal bulks of gold and silver, circles being supposed to vary as the squares of their diameters.

Let D and d be the diameters of a sovereign and a shilling, and T and t their thickness; then by the question $md = n\text{D}$, and $pt = q\text{T}$. Hence $\frac{\text{D}}{d} = \frac{m}{n}$; and $\frac{t}{\text{T}} = \frac{q}{p}$. Also D^2T and d^2t express the relative quantities of matter; that is, the quantity of gold in 1 sov. : quantity of silver in 1s.

$$: : D^2T : d^2t,$$

or,
$$: : \frac{D^2}{d^2} : \frac{t}{T}, \text{ or } : : \frac{m^2}{n^2} : \frac{q}{p}.$$

Now the value of a unit of gold × the number of units in 1 sov. : value of a unit of silver × number of units in 1s. :: 20 : 1.

∴ value of unit of gold : value of unit of silver : : 20 × number of units in 1s. : 1 × number of units in 1 sov.

that is : : $20\frac{q}{p} : 1 \times \frac{m^2}{n^2}$, or finally : : $20n^2q : m^2p$; a ratio from which particular values may be deduced by substitution.

EXERCISES.

1. Four numbers are in G. P.; the sum of the first and second is 28, and the sum of the third and fourth 252. Required the numbers.

2. Divide £210 among three persons, the shares being in G. P., and the first having £90 more than the third.

3. Of three continual proportionals the sum is 14, and the sum of the first and second is to the sum of the second and third as 1 to 2. Required the numbers.

4. The sum of four numbers in G. P. is 30, and the last term divided by the sum of the means is $\frac{4}{3}$. Required the numbers.

5. There are three numbers in A. P.; their sum is 21, and the sum of the first and second is to the sum of the second and third as 3 to 4. Find the numbers.

6. Three numbers are in G. P; the sum of the first and second is 9, and the sum of the first and third 15. Find the numbers.

7. How often does the hammer of a common clock, keeping correct time, strike the bell in 30 days?

8. There are three numbers in G. P., such that their product is 64, and the sum of their cubes 584. Required the numbers.

9. The base of a right-angled triangle is 6, and the sides are in A. P.; find them.

10. There are three numbers in G. P., whose sum is 21, and the sum of their squares 189; find them.

11. Four numbers form an A. P., their sum is 32, and the sum of their squares 276. Required the numbers.

12. Suppose 47 articles to be paid for as follows, 1 shilling for the first; 2 for the second; 3 for the third, and so on; what will be the price of the whole?

13. If 6d. be given for the first acre of an estate, 9d. for the second, 1s. for the third, and so on, the last acre being valued for £100; find the number of acres in the estate.

14. In how many days will two travellers meet, of whom one goes 3 miles the first day, 5 the second, 7 the third, and so on; and the other 4 miles the first day, 6 the second, 8 the third, and so on; the whole distance being 168 miles?

Again suppose the distance to be 23661 miles, and that one of them goes regularly 20 miles a day; while the other travels 1 mile the first day, 2 the second, 3 the third, and so on; what will be the number of days, and the distance travelled by each?

15. Suppose one person to travel at this rate; 1 mile the first day, 2 the second, 3 the third, and so on; and 5 days after he started, another person to start, travelling 12 miles a day regularly; how long, and what distance, must the latter travel to overtake the former?

16. Suppose A to travel regularly at the rate of 4 miles an hour, and B to start $2\frac{3}{4}$ hours after him, travelling the

first hour 4½ miles, the second 4¾ miles, the third 5 miles, and so on. In how many hours will he overtake A?

17. The sum of three numbers in H. Proportion is 13, and the product of their extremes is 18. Required the numbers.

18. The A. M. of two numbers exceeds their G. M. by 5, and the G. M. exceeds the H. M. by 4. Required the numbers.

19. Three numbers form an H. P.; the sum of the third with double the first is equal to three times the second, and the sum of the squares of the first and third is equal to 180. Required the numbers.

20. Divide £700 among 4 persons, in shares forming a G. P., and let the difference between the greatest and least terms be to the difference between the means as 37 is to 12.

21. If a person begin trade with £1000; and if he add to his capital £100 the first year, £200 the second, £300 the third, and so on during the first 20 years; and if he then increase it by £200 the 21st year, £400 the 22d year, £600 the 23d year, and so on; what will he be worth at the end of 40 years?

22. An engine without a train can go 24 miles an hour, and its speed is diminished by a quantity which varies as the square root of the number of waggons attached; with 4 waggons its speed is 20 miles an hour. Required the greatest number of waggons which the engine can move.

23. A number consists of three digits, which are in A. P.; the number divided by the sum of its digits is 26, but if 198 be added to it, the digits will be inverted. Required the number.

24. The interior angles of a rectilineal figure are in A. P.; the least angle is 120°, and the common difference 5°. Required the number of sides in the figure.

25. Suppose a body to move for ever in this manner;

namely, 20 miles the first minute, 19 miles the second, $18\frac{1}{20}$ the third, and so on, in G. P. Required the utmost distance it can reach.

26. Two detachments, distant from one another 37 leagues, set out at different times to occupy an important position half way between them; the first increases its march each day by a league and a half, the second by two leagues; they arrive together, the first after 5 days' march, the second after 4. Find the successive days' marches of each detachment.

27. Suppose a person were to offer to pay for 20 horses in this way; a farthing for the first, three farthings for the second, and so on, trebling each time; what would the price amount to?

28. How much ground does a person pass over in collecting 200 stones placed in a straight line, 2 feet apart; supposing that he brings each stone singly to a basket placed on the same line 20 yards from the first stone, and that he starts each time from the spot where the basket stands?

29. A sphere of metal has a hollow space about its centre in the form of a concentric sphere, and its weight is $\frac{7}{8}$ths of the weight of a solid sphere of the same substance and radius; compare the inner and outer radii, having given that the weights of spheres of like substance vary as the cubes of their radii.

30. The value of diamonds varies as the square of their weight, and the square of the value of rubies as the cube of their weight. Suppose now that a diamond of a carats is worth m times the value of a ruby of b carats, and that both together are worth £c. Required the value of a diamond and of a ruby, each weighing n carats.

XXIII. SERIES.

218. We have already seen (Art. 81) that the *sum* of an infinite series is the *limit* towards which we can approximate as nearly as we please, by taking continually more and more of its terms.

219. A *convergent* series is one whose sum cannot exceed some finite quantity, however great the number of terms may be; that is, one which tends to some such quantity as its limit. All other series are said to be *divergent.*

Thus, the series $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27}$, &c., is convergent, tends continually towards $\frac{3}{4}$ as its limit; and only becomes equal to $\frac{3}{4}$ when the number of terms is infinite; while the series $1 + 2 + 3 + 4$, &c., is divergent; the sum of its terms may surpass any quantity however great.[*]

220. A *recurring* series is one in which each term, after a certain number, may be derived from the one before it, by using a constant multiplier.

Thus, a geometrical progression, after the first term, is a recurring series: and the series of Arts. 81, 83, in which $\frac{x}{a}$, $\frac{b}{a}$, &c., are the constant multipliers, are recurring series. Such constant multiplier is called the *scale of relation* of the terms.

221. The series which have been already considered under Progression are generated according to certain fixed laws,

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[*] It would lead into too great detail to enter here into a consideration of the criteria by which to determine whether a given series be convergent or divergent.

from which formulas expressing their sums have been readily deduced, except in the case of an harmonic series. But there is a vast variety of series, governed by different laws, so that no general expression can be obtained for their sum; and recourse must be had to various contrivances, applicable to particular classes. Some of these will now be briefly considered.

1.—METHOD OF DIFFERENCES.

222. Let the terms of any series be a, b, c, d, e, &c., and let the differences of the terms be taken; we shall then have

$$b-a,\ c-b,\ d-c,\ e-d,\ \&c. \quad . \quad . \quad . \quad . \quad (A)$$

If we take again the differences of these, we shall have

$$c-2b+a,\ d-2c+b,\ e-2d+c,\ \&c. \quad . \quad . \quad (B)$$

Proceeding with these in the same manner, we get

$$d-3c+3b-a,\ e-3d+3c-b,\ \&c. \quad . \quad . \quad (C)$$

and so on. Here the series, (A), (B), (C), are called the *first order* of differences, *second order* of differences, and so on. In these, it is manifest that the coefficients are generated in the same way as those of an expanded binomial (Art. 91), the successive subtractions being in fact additions with contrary signs.* Hence, it is manifest that the general expression for the nth order of differences will be

$$\pm a \mp nb \pm \frac{n(n-1)}{1.2}c \mp \frac{n(n-1)(n-2)}{2.3}d,\ \&c. \quad . \quad . \quad (1)$$

in which the *upper* sign is to be taken when n is *even*, and the *lower* when n is *odd*.

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* Thus, if we take the detached coefficients of the first terms of the first and second orders of differences, and perform the multiplication, we have $(1-1)(1-1) = 1-2+1$; and $(1-2+1)(1-1) = 1-3+3-1$; $(1-3+3-1)(1-1) = 1-4+6+1$, &c., for the successive coefficients, these latter being the coefficients of the fourth order of differences, and of the fourth power of a binomial.

Thus, the first term of the fifth order of differences of the series 1, 3, 9, 27, 81, &c., is found to be 32, by substituting these numbers for a, b, c, &c., and 5 for n, the lower series being used as n is odd. This will be verified by actually taking the differences; thus—

	1,	3,	9,	27,	81,	243,
1st order,	2	6	18	54	162	
2d order,	4	12	36	108		
3d order,	8	24	72			
4th order,	16	48				
5th order,	32					

223. To find any term of a series, let the first terms of the several orders of differences (as 2, 4, 8, 16, in the above example), be represented by d_1, d_2, d_3, &c., so that, d_1, $= b - a$, $d_2 = c - 2b + a$, &c., then we shall have by transposition the following values:—

$$\text{(I)} \qquad \begin{aligned} b &= a + d_1 \\ c &= -a + 2b + d_2 \\ d &= a - 3b + 3c + d_3 \\ e &= -a + 4b - 6c + 4d + d_4, \text{ \&c.} \\ f &= a - 5b + 10c - 10d + 5e + d_5. \end{aligned}$$

In these, substitute in the second the value of b taken from the first; in the third, the values of b and c from the two preceding, and so on; we thus get the values,

$$\text{(II)} \qquad \begin{aligned} b &= a + d_1 \\ c &= a + 2d_1 + d_2 \\ d &= a + 3d_1 + 3d_2 + d_3 \\ e &= a + 4d_1 + 6d_2 + 4d_3 + d_4 \\ f &= a + 5d_1 + 10d_2 + 10d_3 + 5d_4 + d_5, \text{ \&c.} \end{aligned}$$

Expressions from which the series may be constructed when its first term, and the first terms of the orders of differences, If now, these coefficients be compared with those of an expanded binomial (Art. 91), it will be obvious that the binomial which has the same coefficients, is one whose index is less by a unit than the number expressing the *place* of any term in the series. Thus, e is the *fifth* term in the series, and the coefficients in its value are those of the *fourth* power of a binomial; the *sixth* term is f, and its coefficients are those of the *fifth* power of the binomial, and so on. Hence, the $(n+1)$th term would have the form of the nth power of the binomial, that is,

$$a + nd_1 + \frac{n(n-1)}{1 \,.\, 2}d_2 + \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3}d_3, \text{ \&c.,}$$

The nth term of the series therefore will be

$$a + (n-1)d_1 + \frac{(n-1)(n-2)}{1 \,.\, 2}d_2 + \frac{(n-1)(n-2)(n-3)}{1 \,.\, 2 \,.\, 3}d_3 + \text{\&c.} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (2)$$

Thus, if the twelfth term of the series $1^3 + 2^3 + 3^3 + 4^3 +$ &c., be required, we know it must be $12^3 = 1728$. To find it by the above series, we have $a = 1$, $n = 12$; then to obtain d_1, d_2, &c., we take the differences as follows:

$$\begin{array}{ccccccccc}
1 & & 8 & & 27 & & 64 & & 125 \\
& 7 & & 19 & & 37 & & 61 & \\
& & 12 & & 18 & & 24 & & \\
& & & 6 & & 6 & & & \\
& & & & 0 & & & &
\end{array}$$

Hence, $d_1 = 7$, $d_2 = 12$, $d_3 = 6$, $d_4 = 0$.

Therefore, by substitution in (2) the series gives

$$1 + 77 + 660 + 990 = 1728.$$

224. To find the sum of n terms of a series, assume a series such that its first differences shall be the terms of the series, a, b, c, d, &c., this series will plainly be

$$0,\ a,\ a+b,\ a+b+c,\ a+b+c+d, \text{ \&c.}$$

Then the $(n+1)$th term of this series will be the sum of n terms of the series, a, b, c, d, &c., and its second, third, &c. orders of differences, are the first, second, third, &c., orders of differences of the series a, b, c, d, &c.; that is, a, b, c, d, &c. are the first differences; $b-a$, $c-b$, $d-c$, the *second*, but the *first* in the other series, and so on. And hence the term which is d_1 in the assumed series is a in the other; that which is d_2 in the assumed, is d_1 in the other, and so on. The first terms of the several orders of differences therefore are a, d_1, d_2, d_3, &c. And the $(n+1)$th term, or the sum of n terms of the series, a, b, c, d, &c., therefore is

$$s = na + \frac{n(n-1)}{1.2}d_1 + \frac{n(n-1)(n-2)}{1.2.3}d_2 + \&c. \quad . \quad . \quad . \quad (3)$$

EXAMPLES.

1. Find the sum of n terms of the series of odd numbers.

Here $a=1$, $d_1=2$, $d_2=0$; substituting which in the last formula (3), we easily find $s=n^2$, as already found by A. P., Ex. 5, p. 242.

2. Find the sum of n terms of the series $1^2+2^2+3^2+4^2+$ &c.

Here, $a=1$, and the first terms of the successive differences are $d_1=3$, $d_2=2$, $d_3=0$; substituting which in formula (3), we find, after some reductions,

$$1^2+2^2+3^2+4^2+ \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{2.3}.$$

3. Find the sum of n terms of the series $1^3+2^3+3^3+4^3+$ &c.

Here, Example Art. 223, the differences are $d_1 = 7$, $d_2 = 12$, $d_3 = 6$, $d_4 = 0$; also, $a = 1$. Substituting these in the formula for s (3), we find

$$s = n + \frac{7n(n-1)}{2} + 2n(n-1)(n-2) + \frac{n}{4}(n-1)(n-2)(n-3),$$

which, after several reductions, becomes

$$s = \frac{n^4 + 2n^3 + n^2}{4} = \frac{n^2(n+1)^2}{4},$$

which is thus the expression for the sum of the cubes of the natural numbers.

Now this expression is the square of $\frac{n(n+1)}{2}$, which is the sum of n terms of the natural numbers (A. P. Ex. 6, p. 243); therefore,

$$1^3 + 2^3 + 3^3 + 4^3 + \ldots . + n^3 = (1 + 2 + 3 + 4 + \ldots . + n)^2,$$

that is, the sum of the cubes of n terms of the natural numbers is equal to the square of the sum of n terms of the same numbers.

4. Sum the series 1, 3, 6, 10, 15, 21, 28, &c., to n terms.

The first terms of the differences are 2, 1, 0; also, $a = 1$; the sum is then easily found to be

$$\frac{n(n+1)(n+2)}{2 \, . \, 3}.$$

This series is plainly formed by adding the natural numbers in succession; that is, 1 to 0, 2 to 1, 3 to 3, 4 to 6, 5 to 10, 6 to 15, and so on; and is the same as

$$1 + (1 + 2) + (3 + 3) + (6 + 4) + (10 + 5),$$

or as $1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) + \&c.$

And the following curious relation exists between it and the series of the last examples. Thus,

$$2^2 = 3 + 1 ;\ 3^2 = 6 + 3 ;\ 4^2 = 10 + 6 ;\ 5^2 = 15 + 10, \&c. ;$$

the sum of every consecutive pair, 1 and 3, 3 and 6, 6 and 10, &c., being a square number. Also,

$$2^3=3^2-1^2=(3-1)^3;\ 3^3=6^2-3^2=(6-3)^3;\ 4^3=10^2-6^2=(10-6)^3,\ \&c.^*$$

2.—METHOD BY INDETERMINATE COEFFICIENTS.

225. The following simple method of finding the sum of an infinite series will be found of very general application. It presupposes a knowledge of the *n*th term; but this can in most cases be readily ascertained by inspection, or by the rule for finding the *n*th term of an arithmetic series. It is always applicable when the *n*th term can be expressed by a series of positive and integral powers of *n*. The method will be best seen in a few examples.

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* These properties are easily applied in forming a table of squares and cubes. But a readier method presents itself when we take the orders of differences of a few terms of a series of known squares and known cubes. Thus, in the series, 1, 4, 9, 16, &c., the second differences are all 2; continue the first differences by adding 2, making the series 7, 9, 11, 13, 15, 17, 19, &c.; then the successive square numbers will be found by adding each preceding square to the next term of this A. P. Thus, $7+9=16$; $16+9=25$; $25+11=36$; $36+13=49$, &c.

In like manner, using the differences of the series of cubes in Art. 223, form an A. P. by taking $d_2=24$, as the first term, and $d_3=6$, as the common difference; this gives 24, 30, 36, 42, 48, &c., for the series. With $d_1=61$ as a first term, form another series by adding the *term in advance* in the former series; this gives 61, 91, 127, 169, 217, 271, &c. Then to each preceding cube add the *term in advance* in this latter series, and the series of cubes will be formed. The work may stand thus;

	24	30	36	42	48	54	60, &c.
$d_3=6$	61	91	127	169	217	271	331, &c.
	125	216	343	512	729	1000	1331, &c.

EXAMPLES.

1. Required the sum of the series $1^3 + 3^3 + 5^3 +$ &c. to n terms.

Since (Art. 206) the nth term of the series $1 + 3 + 5 +$ &c. is $2n - 1$; the nth term of this series will be $(2n - 1)^3$.

Assume

$$1^3 + 3^3 + 5^3 \text{ \&c. } \ldots (2n-1)^3 = An^4 + Bn^3 + Cn^2 + Dn.^* \quad (1)$$

Now since the series is true for all values of n, it will be true when n becomes $n+1$; substituting $n+1$ for n, we have, since the $(n+1)$th term of the series

$$1 + 3 + 5 \text{ \&c. is } 2n + 1,$$

$$1^3 + 3^3 + 5^3 + \text{\&c.} \ldots (2n-1)^3 + (2n+1)^3 = A(n+1)^4 + B(n+1)^3 + C(n+1)^2 + D(n+1) \quad . \quad . \quad . \quad (2)$$

Subtracting series (1) from (2), developing the powers and incorporating, we find

$$8n^3 + 12n^2 + 6n + 1 = 4An^3 + (6A + 3B)n^2 + (4A + 3B + 2C)n + (A + B + C + D).$$

By equating the coefficients we deduce the following values, $A = 2$, $B = 0$, $C = -1$, $D = 0$,

which being substituted in (1) give

$$1^3 + 3^3 + 5^3 + \text{\&c.} \ldots (2n-1)^3 = 2n^4 - n^2 = n^2(2n^2 - 1)$$

for the required sum.

2. Required the sum of the series

$$1 \times 3 + 3 \times 5 + 5 \times 7 \ldots . \text{ to } n \text{ terms.}$$

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[*] The reason of this assumption is manifest; there can be no term which does not contain n: for, if there were, it would remain when $n = 0$, but this is impossible, as the series then becomes 0. Nor can there be a higher power than n^4; because the highest power in the greatest or nth term is n^3, therefore, n being the number of terms, the sum of the series must be less than $n(2n - 1)^3$, the highest power in which is n^4.

The first factor of each of these being the terms of the series 1, 3, 5, &c., the first factor of the last term will be as before $(2n-1)$: the second factor evidently exceeds the first by 2; the last term therefore is $(2n-1)(2n+1)$.

Assume

$$1 \times 3 + 3 \times 5 + 5 \times 7 + \&c. \ldots (2n-1)(2n+1) = An^3 + Bn^2 + Cn;$$

substituting $n+1$ for n we have,

$$1 \times 3 + 3 \times 5 + 5 \times 7 + \&c. \ldots (2n-1)(2n+1) + (2n+1)(2n+3) = A(n+1)^3 + B(n+1)^2 + C(n+1),$$

subtracting, developing, and equating the coefficients, we find

$$A = \tfrac{4}{3},\ B = 2,\ C = -\tfrac{1}{3};$$

which, being substituted, give for the required sum

$$\frac{4n^3}{3} + 2n^2 - \frac{n}{3} = \frac{n}{3}(4n^2 + 6n - 1).$$

Similarly all series of this kind may be summed *by first finding the* n*th term, and assuming for the result a series beginning with a power of* n, *whose index exceeds by unity the highest index in the* n*th term.*

3.—METHOD OF SUBTRACTION.

226. Certain arithmetical series may be readily summed by the following method, which will be best understood from a few examples.

EXAMPLES.

1. Required the sum of the series

$$\frac{1}{2.5.8} + \frac{1}{5.8.11} + \frac{1}{8.11.14} + \&c. \text{ to infinity.}$$

Omitting the last factor from the denominator of each term, assume

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11}\ \&c. = m;$$

hence, by transposition

$$\frac{1}{5.8}+\frac{1}{8.11}+\frac{1}{11.14}+\text{\&c.}=m-\frac{1}{2.5},$$

subtracting these equations from each other, term by term, we have

$$\frac{6}{2.5.8}+\frac{6}{5.8.11}+\frac{6}{8.11.14}+\text{\&c.}=\frac{1}{10};$$

therefore

$$\frac{1}{2.5.8}+\frac{1}{5.8.11}+\frac{1}{8.11.14}+\text{\&c.}=\frac{1}{60}.$$

To find the sum of *n* terms is somewhat more difficult, the method is as follows:

As the *n*th term is evidently equal to

$$\frac{1}{(3n-1)(3n+2)(3n+5)},$$

the factors of the denominators differing by 3; we begin by assuming

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\text{\&c.}\;\ldots\ldots\ldots\ldots\;\frac{1}{(3n-1)(3n+2)}+$$
$$\frac{1}{(3n+2)(3n+5)}=m\;\therefore\;\frac{1}{5.8}+\frac{1}{8.11}+\frac{1}{11.14}+\text{\&c.}\;\ldots\ldots$$
$$\ldots\ldots\ldots\;\frac{1}{(3n+2)(3n+5)}=m-\frac{1}{2.5}$$

Proceeding as before we find

$$\frac{6}{2.5.8}+\frac{6}{5.8.11}+\frac{6}{8.11.14}+\text{\&c.}\;\ldots\ldots\ldots\ldots$$
$$\frac{6}{(3n-1)(3n+2)(3n+5)}+\frac{1}{(3n+2)(3n+5)}=\frac{1}{10};\;\therefore\;\frac{6}{2.5.8}$$
$$+\frac{6}{5.8.11}+\text{\&c.}\;\ldots\ldots\;\frac{6}{(3n-1)(3n+2)(3n+5)}=\frac{1}{10}-$$
$$\frac{1}{(3n+2)(3n+5)};\;\therefore\;\frac{1}{2.5.8}+\frac{1}{5.8.11}+\text{\&c.}\;\ldots\ldots$$

$$\frac{1}{(3n-1)(3n+2)(3n+5)}=\frac{1}{60}-\frac{1}{6(3n+2)(3n+5)}.$$

If in this result we make n infinite we find, as before, the sum of the series to infinity to be $\frac{1}{60}$.

It is necessary, in finding the sum of n terms, to continue the first series to $n+1$ terms, in order that it may coincide with the second series, except in the first term; and the second series, being formed by rejecting the first factor from the denominator of each term, must end in

$$\frac{1}{(3n+2)(3n+5)}.$$

2. Sum the series

$$\frac{2}{1.3.5.7}+\frac{3}{3.5.7.9}+\frac{4}{5.7.9.11}+\text{\&c. to infinity.}$$

As before assume

$$\frac{2}{1.3.5}+\frac{3}{3.5.7}+\frac{4}{5.7.9}+\text{\&c. to infinity}=m;$$

$$\therefore\ \frac{2}{3.5.7}+\frac{3}{5.7.9}+\frac{4}{7.9.11}+\text{\&c. to infinity}=m-\frac{2}{1.3.5}-\left(\frac{1}{3.5.7}+\frac{1}{5.7.9}+\frac{1}{7.9.11}+\text{\&c. to infinity.}\right)$$

The second equation is obtained by observing that the numerator of each term is less by 1 than the numerator of that term in the upper series, which has the same denominator; consequently we have to subtract the series included in the vinculum from both sides. Subtracting as before, the lower series from the upper we have

$$\frac{12}{1.3.5.7}+\frac{18}{3.5.7.9}+\frac{24}{5.7.9.11}+\text{\&c.}=\frac{2}{15}+$$

$$\left(\frac{1}{3.5.7}+\frac{1}{5.7.9}+\frac{1}{7.9.11}+\text{\&c. to infinity.}\right)$$

This latter series may be found, as in the last example, $=\frac{1}{60}$.

$$\therefore\ \frac{12}{1.3.5.7}+\frac{18}{3.5.7.9}+\frac{24}{5.7.9\ 11}=\frac{2}{15}+\frac{1}{60}=\frac{3}{20},$$

$\therefore$ dividing both sides by 6, we find

$$\frac{2}{1.3.5.7}+\frac{3}{3.5.7.9}+\frac{4}{5.7.9.11}+\text{\&c.}=\frac{1}{40}.$$

The sum of n terms may be found as before.

The rule may be briefly stated as follows: *Assume a series* = m, *omitting the last factor from the denominator of each term; from it subtract another series, formed by omitting the first factor from the denominator of each term of the original series; and then proceed as in the above examples.*

227. The following general expressions will render the method of subtraction more readily applicable in many cases.

Since $$\frac{n}{p}-\frac{n}{(p+q)}=\frac{nq}{p(p+q)} \quad \text{(Art. 74)}$$

therefore conversely $$\frac{n}{p(p+q)}=\frac{1}{q}\left(\frac{n}{p}-\frac{n}{p+q}\right);$$

an expression by which we can find two fractions whose difference shall be equal to any given fraction; so that if there be a series of fractions of the form $\frac{n}{p(p+q)}$, the sum of the series will equal $\frac{1}{q}$th the difference between a series of the form $\frac{n}{p}$ and another of the form $\frac{n}{p+q}$.

Thus, let it be required to find the sum of the series

$$\frac{1}{3.5}+\frac{1}{4.6}+\frac{1}{5.7}+\frac{1}{6.8}\ \text{\&c.}$$

we have for the general term of the series

$$\frac{1}{(n+2)(n+4)} \text{ or } \frac{1}{2}\left(\frac{1}{n+2}-\frac{1}{n+4}\right)$$

q being 2 and p, $n+2$. The successive terms will be obtained by taking n equal to 1, 2, 3, &c., and for the last but one, last but two, &c., changing n into $n-1$, $n-2$, &c., We then have for the series

$$\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right),\ \frac{1}{2}\left(\frac{1}{4}-\frac{1}{6}\right) \text{ \&c.};$$

or arranging the positive and negative terms in different lines, the series may be thus expressed:

$$S=\frac{1}{2}\left\{\begin{array}{l}\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots+\frac{1}{n+2} \\ \qquad\quad -\frac{1}{5}-\frac{1}{6} \qquad\quad -\frac{1}{n+2}-\frac{1}{n+3}-\frac{1}{n+4}\end{array}\right\}$$

$$\text{or } S=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}-\frac{1}{n+3}-\frac{1}{n+4}\right)=\frac{7}{24}-\frac{2n+7}{2(n+3)(n+4)}.$$

The sum to infinity is plainly $\frac{7}{24}$.

For series, the factors of whose denominators are equidifferent, the following general formula, deduced as above, will give the sum—

$$\frac{1}{p_1p_2p_3\cdots p_n}=\frac{1}{p_n-p_1}\left(\frac{1}{p_1p_2p_3\cdots p_{n-1}}-\frac{1}{p_2p_3p_4\cdots p_n}\right)$$

4.—PILING OF BALLS.

228. As an additional illustration of the above theory, the method of finding the number of balls in a pile may be adduced. These piles are either *square, triangular*, or *rectangular;* the balls being all of equal size in any one pile.

The square pile is a square pyramid, and each layer of balls is a square whose side exceeds that of the one above it by one ball. Hence, the number of layers is the same as the number of balls in one side of the lowest layer: on the top there is evidently only one ball. It is therefore plain that the number of balls will be the sum of n terms of the series $1^2 + 2^2 + 3^2 + \&c. \ldots n^2$, n being the number of layers. This sum (Art. 224), or found by the preceding method, is $\frac{n(n+1)(2n+1)}{2 . 3}$.[*]

229. In the triangular pile the base, and consequently each layer, is an equilateral triangle, whose side exceeds that of the one above it by one ball, there being plainly only one

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[*] This is the general formula for the sum of the square numbers, or third order of *polygonal numbers*. These, and the other orders which are called *linear*, *triangular*, *pentagonal*, &c., are derived from the A. P., whose first term is 1, and difference d. The sum of this series is $n + \frac{1}{2}n(n-1)d$; in which, if we put for d, 0, 1, 2, 3 . . . successively, we obtain expressions which are called the 1st, 2d, 3d, 4th, &c. orders of *polygonal numbers*. These are as follows:

1st order, nth term	n;	1, 2, 3, 4, 5 . . .	*linear.*
2d „	„ $\frac{1}{2}n(n+1)$;	1, 3, 6, 10, 15 . . .	*triangular.*
3d „	„ n^2;	1, 4, 9, 16 . . .	*square.*
4th „	„ $\frac{1}{2}n(3n-1)$;	1, 5, 12, 22 . . .	*pentagonal.*

$S = \frac{1}{2}n(n+1)$ is the expression for the sum of the numbers of the 1st order; that of the second is given at the end of next Art. The names were orignally given from the way in which the numbers of the different kinds admit of being arranged, if the units in each be represented by dots. The piling of balls affords an illustration.

The *figurate* numbers form a series of different orders, the nth term of any order being the sum of n terms of the preceding;

1st order	1,	1,	1,	1,	1	4th order	1,	4,	10,	20,	35
2d „	1	2	3	4	5	5th „	1	5	15	35	70
3d „	1	3	6	10	15	6th „	1	6	21	56	126
						&c.				&c.	

where $5 = 1 + 1 + 1 + 1 + 1$; $15 = 1 + 2 + 3 + 4 + 5$, &c.; in general the nth term of the 2d order is n; of the third, $\frac{n(n+1)}{1 . 2}$; of the 4th, $\frac{n(n+1)(n+2)}{1 . 2 . 3}$ &c.

ball on the top. Each of the horizontal layers consists of a number of rows, regularly decreasing by unity from the number in the outside row down to 1. Hence it is plain that the number in the nth layer will be the sum of the series,

$$\ldots\ldots\ldots\ 1+2+3+\ \ldots\ldots\ldots\ n=\frac{n(n+1)}{2}.$$

$$\ldots\ldots\ldots\ (n-1)\text{th}\ \ldots\ 1+2+3+\ \ldots\ (n-1)=\frac{(n-1)n}{2}$$

$$\ldots\ldots\ (n-2)\text{th}\ \ldots\ 1+2+3+\ldots(n-2)=\frac{(n-2)(n-1)}{2}$$

&c. &c. &c.

In the second layer the number will be $1+2\ \ldots\ =\frac{2\times 3}{2}$.

$\ldots\ldots$ first $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 1\ \ldots\ =\frac{1\times 2}{2}$.

Hence the series we have to sum is

$$\frac{1\times 2+2\times 3+\ \ldots\ldots\ (n-1)n+n(n+1)}{2};$$

where n denotes the number of layers, or, which is the same thing, the number of balls in one side of the bottom layer. The sum of this series will be readily found to be

$$\frac{n(n+1)(n+2)}{2\,.\,3}.$$

230. In the rectangular pile, each layer is a rectangle, whose sides respectively exceed the sides of the one above it by one ball, the number of layers being evidently the same as the number of balls in the shorter side of the base. Let n = this number, and $n+m$ = the number on the longer side, where m denotes the difference, which manifestly remains the same in all the layers. The top layer is a single row of balls, but for the sake of uniformity it may be considered a rect-

angle, whose sides are 1 and $1 + m$. The number of balls in the bottom layer is $n(n + m)$; in the next, $(n - 1)(n + m - 1)$, and so on to the top, which is $1(1 + m)$: hence the series we have to sum is

$$1(1 + m) + 2(2 + m) + 3(3 + m) + \ldots\ldots\ n(n + m).$$

The simplest method of doing this is the following: taking off the brackets it becomes

$$\begin{aligned} &1^2 + 2^2 + 3^2 + \&c. \ldots\ldots\ldots\ldots + n^2 \\ &+ m(1 + 2 + 3 + \&c. \ldots\ldots\ldots\ldots\ n). \end{aligned}$$

The former series has been already shewn

$$= \frac{n(n + 1)(2n + 1)}{6},$$

and the latter $= \dfrac{n(n + 1)}{2}$: hence the sum is

$$\frac{n(n + 1)(2n + 1)}{6} + m\frac{n(n + 1)}{2} = \frac{n(n + 1)}{6}(2n + 1 + 3m);$$

or if instead of $n + m$, the longest side of the base, we write l, we have

$$\frac{n(n + 1)(3l - n + 1)}{6}.$$

If $l = n$, the pile becomes square, and we have the same result as before,

$$\frac{n(n + 1)(2n + 1)}{6}.$$

We might have obtained the above result in another manner; for if we cut off n rows from the longer side of the base, by a line terminating at the first ball on the top, we shall have a square pile of n layers, whose sum is

$$\frac{n(n + 1)(2n + 1)}{6},$$

and m triangles reclining on it, each consisting of n rows of balls, whereof n is the number in the longest, and 1 in the shortest side; or in other words, each triangle is the sum of the series,

$$1+2+3+\&c.\ldots n=\frac{n(n+1)}{2},$$

and as there are m of these triangles, the sum of them all is $m\,\frac{n(n+1)}{2}$, and therefore the number of balls in the whole pile is, as before, equal to

$$\frac{n(n+1)(2n+1)}{6}+m\,\frac{n(n+1)}{2}=\frac{n(n+1)(3l-n+1)}{6}.$$

231. Should the pile be incomplete, the easiest method of finding the number of balls is, first, to find the number in the complete pile, and afterwards to find the number in the wanting pile, and to subtract the one sum from the other. Thus, if it were proposed to find the number of balls in an incomplete triangular pile of 21 layers, one side of the bottom layer being 40; we first find the number there would be in the pile, if complete, which is found by the formula,

$$\frac{n(n+1)(n+2)}{6}=\text{(in this case)}$$

$$\frac{40\times 41\times 42}{6}=40\times 41\times 7=11480:$$

the number of layers ought to be 40, but by hypothesis there are only 21, therefore there are 19 wanting, consequently the number of balls wanting to complete the pile is

$$\frac{19\times 20\times 21}{6}=19\times 10\times 7=1330.$$

Hence the number of balls in the incomplete pile is

$$11480-1330=10150.$$

5.—RECURRING SERIES.

232. The scale of relation (Art. 220) in series of this kind must be known before we can find the sum, or original fraction from whose expansion the series has arisen.

Let $1 + 5x + 9x^2 + 13x^3 + 17x^4 + 21x^5$, &c.

be such a series. To find the scale of relation by a general process, suppose it to be a, a'; then $9x^2 = ax \,.\, 5x + a'x^2 \,.\, 1 = (5a + a')x^2 \;\therefore\; 5a + a' = 9$; also, $13x^3 = ax \,.\, 9x^2 + a'x^2 \,.\, 5x = (9a + 5a')x^3 \;\therefore\; 9a + 5a' = 13$. These two equations between a and a' give $a = 2$, and $a' = -1$; so that the scale of relation of the terms is $2x, -x^2$; or, since it is the coefficients only in the series that are required, the powers being known, the scale is $2, -1$, or $2 - 1$. A little consideration would in fact have shewn us that these are the multipliers by which the successive terms are obtained. For, $9x^2 = 2x.5x - x^2 \,.\, 1$; $13x^3 = 2x \,.\, 9x^2 - x^2 \,.\, 5x$; $17x^4 = 2x.13x^3 - x^2 \,.\, 9x^2$; $21x^5 = 2x.17x^4 - x^2.13x^3$, &c. Also, it is manifest from this relation of the terms, that three of them taken with their proper signs are equal to zero $(21 - 34 + 13 = 0;\ 17 - 26 + 9 = 0$, &c.); in the same manner as in a geometrical series $A_n - rA_{n-1} = 0$. In such series as the above, this relation is denoted by the general expression $A_n - 2xA_{n-1} + x^2A_{n-2} = 0$. In the following series, $2 + 4x + 14x^2 + 46x^3 + 152x^4 +$ &c., $152x^4 = 3x \,.\, 46x^3 + x^2 \,.\, 14x^2 \;\therefore\; 152x^4 - 3x \,.\, 46x^3 - x^2 \,.\, 14x^2 = 0$: or, generally, the law connecting the terms is

$$A_n - 3xA_{n-1} - x^2A_{n-2} = 0$$

in which three coefficients are the scale of relation, $1 - 3x - x^2$. In the geometrical series above it is $1 - r$.

233. The sum of n terms of a recurring series may be found as follows :

Let $S = A + A_1x + A_2x^2 + A_3x^3 \ldots + A_{n-1}x^{n-1}$.

Then the scale of relation being a, a', or $a - a'$,

$$axS = Aax + A_1ax^2 + A_2ax^3 \ldots + A_{n-2}ax^{n-1} + A_{n-1}ax^n$$

and

$$a'x^2S = \qquad Aa'x^2 + A_1a'x^3 \ldots + A_{n-3}a'x^{n-1} + A_{n-2}a'x^n + A_{n-1}a'x_{n+1}$$

Hence

$$S - axS - a'x^2S = A + A_1x - Aax - A_{n-1}ax^n - A_{n-2}a'x^n - A_{n-1}a'x^{n+1}$$

the other members on the right side vanishing, in consequence of the relation of three consecutive terms pointed out above. Hence we have

$$S = \frac{A + x(A_1 - aA) - x^n\{aA_{n-1} + a'A_{n-2} + a'xA_{n-1}\}}{1 - ax - a'x^2} \quad . \ (1)$$

Now, since x is supposed to be less than unity, as n increases without limit, the quantity into which it enters as a factor in the numerator will decrease without limit, and will vanish when n is infinite. The sum of an infinite number of terms of a recurring series therefore is

$$S = \frac{A + x(A_1 - aA)}{1 - ax - a'x^2} \quad . \ . \ . \ . \ . \ (2)$$

Similarly, if the scale of relation were a, a', a'', we should have for the sum

$$S = \frac{A + A_1x + A_2x^2 - ax(A + A_1x) - a'x^2A}{1 - ax - a'x^2 - a''x^3}\ ^* \quad . \ (3).$$

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[*] It will sometimes happen that the scale of relation thus determined will not produce the succeeding terms; that is, will not satisfy the equations of the higher degrees; the quantities found cannot therefore be the true scale, and other equations must be tried. Thus, in Ex. 38, if we proceed in the usual way, we shall find two values, $3\frac{1}{7}$, $-3\frac{2}{7}$, which will

EXAMPLE.

Required the scale of relation and the sum of the recurring series $1 + 2x + 8x^2 + 28x^3 + 100x^4 + 356x^5 + \&c.$

Here, equating the terms as before, the scale being assumed as a, a', we have the equation for determining these two unknowns,

$$\begin{aligned} 8a + 2a' &= 28 \\ 2a + a' &= 8 \end{aligned}$$

whence $a = 3$, and $a' = 2$; the scale therefore is $3x$, $2x^2$.

Also, $A = 1$, $A_1 = 2$; substituting these four values in equation (2) we find

$$S = \frac{1 - x}{1 - 3x - 2x^2}.$$

6.—REVERSION OF SERIES.

234. The process called the *reversion of series* consists in developing an unknown quantity contained in a series, in terms of another to which the series is given as equal. It is only an application of the method of indeterminate coefficients, and has been sufficiently illustrated already. The expression there given, Ex. 2,

$$y = \frac{x}{a} - \frac{bx^2}{a^3} + \frac{2b^2 - ac}{a^5}x^3 - \frac{5b^3 - 5abc + a^2d}{a^7}x^4 \ \&c.,$$

is a general formula applicable to all such developments.

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not satisfy the equation, $9^2x^4 = ax \cdot 7^2x^3 + a'x^2.25x^2$; hence, we must assume a new scale, a, a', a'', and form three equations.

In many cases the scale may be found by taking the successive orders of differences. If the nth differences vanish, the scale of relation is had by the expression,

$$n - \frac{n(n-1)}{1.2} + \frac{n(n-1)(n-2)}{1.2.3} - \&c.$$

7.—INTERPOLATION OF SERIES.

235. The process called *interpolation of series* is that whereby a wanting term is introduced at any given place in a series. We can do this if we know that the order of differences expressed by the same number as the number of given terms, is either zero or so small that it may be neglected; that is, if in a series of $n + 1$ consecutive terms, one is wanting, and it is known that the nth order of differences vanishes, the term may be interpolated.

Let it be required to find the third term of the series 3, 40, ... 576, 1375, 2808, the fifth differences being supposed to vanish; then from the formulæ I, Art. 223, we have $d_5 = 0 = f - 5e + 10d - 10c + 5b - a.$
Whence by substituting the values of the letters in the given series, we find $c = \dfrac{1890}{10} = 189,$ the wanting term.

This method is applied in many cases in which the differences do not vanish, but in which they present no significant figures of an order high enough to affect the terms to be calculated. *See* Exercises in Logarithms.

EXERCISES.

1. Find the first term of the fifth order of differences of the series, 1, 3, 9, 27, 81, &c.

2. Find the first term of the third order of differences of the series, 1, 5, 15, 35, 70, &c.

3. Let $(2n + 1)n^2$ be the general term of a series; develop the series from the beginning by taking n equal 1, 2, 3, &c., successively, and find the first term of the fourth order of differences.

4. Find the 20th term and sum of 20 terms of the same series.

5. Construct the series whose first term is 2, and the first terms of the first, second, and third orders of differences 4, 2, 0, respectively; and find the 20th term.

6. Let $n(2n+1)^2$ be the general term of a series; develop it; shew that the differences of the third order are constant; find the 100th term, and the sum of 100 terms.

7. Construct the series whose first term is 1, and the first terms of the successive orders of differences, 1, 2, 3, 4, 0, respectively; find the 24th term and the sum of 24 terms.

8. Find the first term of the eighth order of differences of the series 1, 3, 9, 27, 81, &c.

9. Find what relation the nth term of the series 1, 3, 6, 10, 15, 21, &c., bears to the nth term of the series 2, 6, 12, 20, 30, &c.

10. Develop the series whose general term is $(n-1)^2(3n+2)$, and find the sum of 50 terms.

Sum the following series :—

11. $1+4+10+20+30$ &c., to n terms.
12. $1+3+6+10+15+21$ &c., to 100 terms.
13. The odd numbers, to 60 and to n terms.
14. $1^2+2^2+3^2$ &c., and $1^2+3^2+5^2$ &c., to n terms by indeterminate coefficients.
15. $1^4+2^4+3^4$ &c., and $1^5+2^5+3^5+$ &c., to n terms by indeterminate coefficients.
16. $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}$, &c., to n terms and to infinity.
17. $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}$ &c., to infinity.
18. Whose general term is $(2n-1)(2n+1)^2$ to n terms.
19. $1.2+2.3+3.4+$ &c., and $1.2.3+2.3.4+3.4.5+$ &c., to n terms.

20. $\frac{1}{2.5.8}+\frac{3}{5.8.11}+\frac{5}{8.11.14}$ &c., to infinity.

21. $\frac{1}{1.3.5.7}+\frac{1}{3.5.7.9}+\frac{1}{5.7.9.11}$ &c., to n terms and to infinity.

22. $\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$ &c., to infinity.

23. $\frac{1}{2.4.6}+\frac{1}{4.6.8}+\frac{1}{6.8.10}$ &c., to infinity.

24. $\frac{1}{2.4.6.8}+\frac{1}{4.6.8.10}+\frac{1}{6.8.10.12}$ &c., to infinity.

25. $\frac{5}{1.2.3}+\frac{6}{2.3.4}+\frac{7}{3.4.5}+\frac{8}{4.5.6}$ &c., to infinity.

26. $\frac{1}{1.3.5.7}+\frac{2}{3.5.7.9}+\frac{3}{5.7.9.11}$ &c., to infinity.

27. $\frac{1}{1.5}-\frac{1}{3.7}+\frac{1}{5.9}$ &c., to infinity.

28. $\frac{5}{1.4}-\frac{7}{2.5}+\frac{9}{3.6}-\frac{11}{4.7}+\frac{13}{5.8}-$ &c., to infinity.

29. $\frac{1}{a(a+x)}+\frac{1}{(a+x)(a+2x)}+\frac{1}{(a+2x)(a+3x)}$, &c., to infinity.

30. $a+2ar+3ar^2+4ar^3+$ &c., to n terms and to infinity.

Find the scale of relation, and the sum to infinity, in the following recurring series :—

31. $1+3x+5x^2+7x^3+9x^4+$ &c.

32. $1+2x+7x^2+20x^3+61x^4+182x^5$, &c.

33. $1+2x+3x^2+4x^3+$ &c.

34. $1-2x+3x^2-4x^3+$ &c.

35. $4+9x+\quad 1x^2+51x^3$, &c.

36. $1 + 6x + 12x^2 + 48x^3 + \&c.$

37. $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \&c.$ Ex. 9, p. 89.

38. $1^2 + 3^2x + 5^2x^2 + 7^2x^3 + 9^2x^4 + \&c.$

39. $1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \&c.$

40. $1 + \frac{3}{x} + \frac{5}{x^2} + \frac{7}{x^3} + \frac{9}{x^4} + \&c.$

41. $1 + 8x + 27x^2 + 64x^3 + 125x^4 + 216x^5 + 343x^6.$

42. $1 . 3 + 3 . 5x + 5 . 7x^2 + 7 . 9x^3 + 9 . 11x^4 + \&c.$

Find the number of balls,—

43. In a triangular pile of 30 layers.

44. In a square pile of 30 layers.

45. In a rectangular pile, the length of the base being 60, and the breadth 30.

46. In the same, the length and breadth of the lowest row being 52 and 34.

47. In a complete triangular pile of 20 layers, and an incomplete rectangular pile of 12 layers, the length and breadth of the base being 40 and 20.

48. In an incomplete square pile, the side of the base being 24, and of the top 8.

49. Shew that the number of balls in the two lowest layers of a complete triangular pile, equals the number in the base of a complete square pile of the same number of layers.

50. There are two complete piles of balls, one triangular and the other square, such that the whole number of balls in the former is to the whole number in the latter as 6 to 11. Required the number of balls in each pile, the bottom row of both having the same number of balls in a side.

XXIV.—LOGARITHMS AND EXPONENTIALS.

236. When the several terms of a series of numbers are considered as powers of some common root, the indices of those powers are called the *Logarithms* of the numbers to which the powers are equal. Thus, if we consider $\frac{1}{16}$, $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, &c. as powers of the common root 2, we shall have

$$\begin{array}{ll} \frac{1}{16} = 2^{-4}, & 2 = 2^{1}, \\ \frac{1}{8} = 2^{-3}, & 4 = 2^{2}, \\ \frac{1}{4} = 2^{-2}, & 8 = 2^{3}, \\ \frac{1}{2} = 2^{-1}, & 16 = 2^{4}, \\ 1 = 2^{0}, & 32 = 2^{5}, \end{array}$$

&c.

where the indices -4, -3, -2, -1, 0, 1, 2, 3, &c. are the logarithms of the numbers $\frac{1}{16}$, $\frac{1}{8}$, $\frac{1}{4}$, &c.

But the same numbers may be considered as powers of the common root 4, and we shall then have

$$\begin{array}{ll} \frac{1}{16} = 4^{-2}, & 4 = 4^{1}, \\ \frac{1}{8} = 4^{-\frac{3}{2}}, & 8 = 4^{\frac{3}{2}}, \\ \frac{1}{4} = 4^{-1}, & 16 = 4^{2}, \\ \frac{1}{2} = 4^{-\frac{1}{2}}, & 32 = 4^{\frac{5}{2}}, \\ 1 = 4^{0}, & 64 = 4^{3}, \\ 2 = 4^{\frac{1}{2}}\,; & \end{array}$$

&c.

where the indices -2, $-\frac{3}{2}$, -1, &c. are the logarithms of the corresponding numbers.

237. If it be required to find what power of 4 is equal to 13, it will appear from the table that since 13 is between 8 and 16, the index of 4, that is, the logarithm of 13, will be between $\frac{3}{2}$ and 2; and, from Arts. 86, 100, that if such an index cannot be exactly found, it may be approximated to. Hence, logarithms may be calculated for all numbers.

238. The common root, of which other numbers are taken as powers, is called the *base* of the system of logarithms. Thus, in the examples of Art. 236, the bases are 2 and 4.

239. Any number may be assumed as the base; and it is plain that every base will give a different system of logarithms: but the base which has been found most convenient is 10. Logarithms constructed to the base 10 are called, for the sake of distinction, *common logarithms*. Hence, the *common logarithm of a number is the index of the power of* 10 *which is equal to that number.* The only other base to which logarithms have ever been constructed, is the number 2·718281828459 . . . This was the base chosen by Baron Napier, the inventor of logarithms; and hence these are called Napierian logarithms. Since $a^0 = 1$ it is plain that, in every system, the logarithm of 1 is 0.

Taking, therefore, 10 for the base, its powers will be as follows:

$$\begin{array}{ll} 10^0 = 1, \text{ and} & 10^{-1} = \frac{1}{10} = {\cdot}1\,; \\ 10^1 = 10, & 10^{-2} = \frac{1}{100} = {\cdot}01\,; \\ 10^2 = 100, & 10^{-3} = \frac{1}{1000} = {\cdot}001\,; \\ 10^3 = 1000, & 10^{-4} = \frac{1}{10000} = {\cdot}0001\,; \\ 10^4 = 10000, & 10^{-5} = \frac{1}{100000} = {\cdot}00001\,; \\ & \&c. \end{array}$$

The several indices are here the common logarithms of the corresponding powers.* It is plain that the logarithms of all numbers between 1 and 10, that is, numbers of one figure, are between 0 and 1; that the logarithms of numbers between 10 and 100, that is, those consisting of two figures, are between 1 and 2; and in general that the integral part of the logarithm, which is usually called the *characteristic*, is always a unit less than the number of digits. Also, it is plain that of decimals between ·1 and ·01, the logarithms are greater than -2, and less than -1, or that they are -2 increased by some fraction; and those of decimals between ·01 and ·001, are -3 increased by some fraction; and, in general, that the logarithm of a decimal consists of a positive fraction combined with a negative integer, greater by 1 than the number of zeros prefixed to the significant figures of the decimal. Thus if $-3 + \cdot726401$ be a logarithm, the corresponding number will be a decimal fraction, having two zeros prefixed to its significant figures. It is usual to write such logarithms with the sign $-$ over the integral part, omitting the sign $+$ before the fractional part. Thus $\bar{3}\cdot726401$ means $-3 + \cdot726401$.* The decimal part of a logarithm is often called the *mantissa*.

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* By reasoning, as in Art. 237, it might be shewn from the above table, that logarithms may be approximated to when the numbers are not exact powers of 10. This will also appear from the following example. Suppose the logarithm of 5 to be required, or from $10^x = 5$ to find x; first, x is plainly greater than $\frac{1}{2}$; let $x = \frac{2}{3}$, then since $10^{\frac{2}{3}} = \sqrt[3]{100}$, and $5 = \sqrt[3]{125}$, $\frac{2}{3}$ is less than log. 5. Let $x = \frac{3}{4}$; then $10^{\frac{3}{4}} = \sqrt[4]{1000}$; but $5 = \sqrt[4]{625}$ $\therefore$ $\frac{3}{4}$ is greater than log. 5; log. 5 is therefore between $\frac{2}{3}$ and $\frac{3}{4}$, or $\frac{8}{12}$ and $\frac{9}{12}$, that is between ·666666 and ·749999. This number is afterwards found to be ·698970. *See Art.* 249.

240. The *modulus* of a system of logarithms is a constant multiplier, by means of which, when the logarithm of a number in one system is known, its logarithm in another system may be found.

241. It follows from Arts. 35, 41, that the logarithm of a product is equal to the sum of the logarithms of its factors; and that the logarithm of a quotient is equal to the difference

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* When we have such a number as $-3 + \cdot726401$, it might appear more natural actually to incorporate the two parts, which would give $-2\cdot273599$; but the mode of notation explained in the text, enables us to make the same logarithm serve for all numbers, integral, fractional, and mixed, which consist of the same figures, as will appear in Art. 242.

of the logarithms of the dividend and divisor: that is, log. $abc = \log. a + \log. b + \log. c$; and log. $\frac{a}{b} = \log. a - \log. b$.

Also, from Arts. 88, 94, it follows, that the logarithm of the nth power of a number is n times the logarithm of the number; and the logarithm of the nth root, the nth part of the logarithm of the number: or that log. $b^n = n \log. b$; log. $b^{\frac{1}{n}} = \frac{1}{n} \log. b$; and log. $b^{\frac{m}{n}} = \frac{m}{n} \log. b$.

242. Hence, it is plain that the subtraction of 1 from any logarithm, gives the logarithm of the quotient obtained by dividing the number by 10; the subtraction of 2, the logarithm of the quotient obtained by dividing the number by 100, &c.; and, therefore, that the logarithms of decimals will differ from the logarithms of integers composed of the same figures, only in the characteristic. It is this property of 10 which makes it the most convenient base.

243. If there be a series of quantities in geometrical progression, their logarithms will be in arithmetical progression.

Let b, cb, c^2b, c^3b, &c., be a geometrical progression; and let x, x', be the logarithms of b and c to the base a, so that $a^x = b$, $a^{x'} = c$: then (Art. 241) the progression is the same as

$$a^x,\ a^{x+x'},\ a^{x+2x'},\ a^{x+3x'},\ \&c.$$

in which the logarithms

$$x,\ x+x',\ x+2x',\ x+3x',\ \&c.$$

are in arithmetical progression.

244. To expand a^x in a series according to the powers of x; assume $a^x = h + kx + lx^2 + mx^3 + nx^4 + \&c.$

Squaring both sides we have

$$a^{2x} = h^2 + 2hkx + \begin{array}{r|r|r|l} k^2 & x^2 + 2hm & x^3 + l^2 & x^4 + \&c. \\ +\,2hl & +\,2kl & +\,2km & \\ & & +\,2hn & \end{array} \qquad (1)$$

also putting $2x$ for x in the first series we obtain

$$a^{2x} = h + 2kx + 4lx^2 + 8mx^3 + 16nx^4 + \&c. \qquad (2)$$

Hence (Art. 159), equating the coefficients of the like terms, we find

$h^2 = h$, or $h = 1$; $2hk = 2k$ or $k = k$; $k^2 + 2hl = 4l$, or $l = \frac{k^2}{1\,.\,2}$; $2hm + 2kl = 8m$, or $m = \frac{k^3}{1\,.\,2\,.\,3}$.

Substituting in the original series these values of the coefficients, we find

$$a^x = 1 + kx + \frac{k^2x^2}{1\,.\,2} + \frac{k^3x^3}{1\,.\,2\,.\,3} + \frac{k^4x^4}{1\,.\,2\,.\,3\,.\,4}\ \&c. \qquad (3)$$

In this k is yet unknown; to find it, put $a = 1 + b$, and we then have

$$a^x = (1 + b)^x = 1 + xb + \frac{x(x-1)}{1\,.\,2}b^2 + \frac{x(x-1)(x-2)}{1\,.\,2\,.\,3}b^3 + \&c.,$$

or, if the multiplications indicated be performed, and M, N, &c., be put for the coefficients, which x^2, x^3, &c. then have,

$$a^x = 1 + \left\{ b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \&c. \right\} x + \mathrm{M}x^2 + \mathrm{N}x^3 + \&c.$$

As this value of a^x must be identical with the former, we have, by equating the coefficients, and since $b = a - 1$,

$$k = b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \&c. = \frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \&c.$$

If we represent by e that value of a which makes $k = 1$, we shall have

$$e^x = 1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} + \&c.$$

and if in this series we put $x = 1$, we find

$$e = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c.$$

or, by actual calculation of the values of the several terms,

$$e = 2{\cdot}718281828459 \ldots$$

245. A power having an unknown quantity for its index, is called an *exponential* quantity; and the formula, series (3) last Art., giving the development of a^x is usually known by the name of *the exponential theorem*: it is of great importance in analysis.

246. To find the logarithm of a number; let N be any number, and x its logarithm to the base a, so that $a^x = \mathrm{N}$; then it is required to determine x.

Raising both sides to the power whose index is y, we have $$a^{xy} = \mathrm{N}^y.$$

Expanding both sides by the exponential theorem we find

$$1 + \frac{kxy}{1} + \frac{k^2x^2y^2}{1.2} + \&c. = 1 + \frac{k'y}{1} + \frac{k'^2y^2}{1.2} + \frac{k'^3y^3}{1.2.3} + \&c.$$

where

$$k = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \&c.$$

and

$$k' = (\mathrm{N}-1) - \tfrac{1}{2}(\mathrm{N}-1)^2 + \tfrac{1}{3}(\mathrm{N}-1)^3 - \tfrac{1}{4}(\mathrm{N}-1)^4 + \&c.$$

as proved before.

Equating the coefficients of y on both sides of the above equation, we have

$$kx = k', \text{ and } x = \frac{k'}{k}.$$

We have therefore,

$$\log.\ \mathrm{N} = \frac{(\mathrm{N}-1) - \frac{1}{2}(\mathrm{N}-1)^2 + \frac{1}{3}(\mathrm{N}-1)^3 - \frac{1}{4}(\mathrm{N}-1)^4, \&c.}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4, \&c.}$$

The numerator of this fraction is independent of the base a, and will be the same in every system of logarithms ; but the value of the denominator will depend upon a, and will be different in different systems. It is the reciprocal of this denominator which is called the *modulus* of the system : putting the modulus = M, and $N = 1 + n$, we have

$$\log. (1 + n) = M(n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4, \&c.)$$

and therefore,

$$\log. (1 - n) = M(-n - \tfrac{1}{2}n^2 - \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4, \&c.)$$

Hence, by subtraction, and Art. 241, we have

$$\log. (1 + n) - \log. (1 - n) = \log. \frac{1 + n}{1 - n} = 2M(n + \tfrac{1}{3}n^3 + \tfrac{1}{5}n^5, \&c.)$$

Now, since in this equation n may have any value, let

$$n = \frac{1}{2N + 1} ; \text{ then } 1 + n = \frac{2N + 2}{2N + 1} ;$$

$$\text{and} \qquad 1 - n = \frac{2N}{2N + 1} ; \text{ therefore, } \frac{1 + n}{1 - n} = \frac{N + 1}{N}.$$

Hence we have by substitution and Art. 241, and transposing log. N,

$$\log. (N + 1) = \log. N + 2M\left(\frac{1}{2N + 1} + \frac{1}{3(2N + 1)^3} + \frac{1}{5(2N + 1)^5} + \frac{1}{7(2N + 1)^7}, \&c.\right)$$

This formula will give the logarithm of any number, that of the next less number being known.

247. The modulus M may have any value. In Napierian logarithms, it is taken equal to unity ; for calculating these, therefore, we have the general formula

$$\log. (N + 1) = \log. N + 2\left(\frac{1}{2N + 1} + \frac{1}{3(2N + 1)^3} + \frac{1}{5(2N + 1)^5} + \frac{1}{7(2N + 1)^7}, \&c.\right)$$

in which N must be taken, successively, equal to the natural numbers, 1, 2, 3, 4, &c. We thus find

$$\begin{aligned}
&\log.\ 2 = 2\left(\frac{1}{3} + \frac{1}{3^4} + \frac{1}{5\,.\,3^5} + \frac{1}{7\,.\,3^7}\ \&\text{c.}\right) && = \cdot 693147. \\
&\log.\ 4 = 2\log.\ 2 && = 1\cdot 386294. \\
&\log.\ 5 = 2\left(\frac{1}{9} + \frac{1}{3\,.\,9^3} + \frac{1}{5\,.\,9^5}\ \&\text{c.}\right) + \log.\ 4 && = 1\cdot 609438. \\
&\log.\ 10 = \log.\ 2 + \log.\ 5 && = 2\cdot 302585. \\
&\qquad \&\text{c.} && \qquad \&\text{c.}
\end{aligned}$$

248. The logarithm of a number in one system being given, to find its logarithm in any other system; let N be a given number, and let y, x, be its logarithms in two systems whose bases are a, e; so that

$$a^y = \mathrm{N},\ e^x = \mathrm{N};$$

and let the logarithms of N in the two systems be denoted by $\log._e \mathrm{N}$, and $\log._a \mathrm{N}$ respectively; then $x = \log._e \mathrm{N}$, and $y = \log._a \mathrm{N}$. Now it is plain that

$$e^x = a^y;$$

and therefore, by taking the logarithms of both sides to the base e

$$x = y \log._e a,$$

whence

$$y = \frac{x}{\log._e a};$$

or substituting the above values of x and y,

$$\log._a \mathrm{N} = \frac{\log._e \mathrm{N}}{\log._e a};$$

or

$$\log._a \mathrm{N} = \log._e \mathrm{N} \times \frac{1}{\log._e a}.$$

This equation expresses the constant ratio which the logarithms of N have to each other, in the two systems.

249. Since the only systems in use are the common and Napierian, let $a = 10,\ e = 2{\cdot}718281828459,$

then we have $\log._{a} N = \dfrac{1}{\log._{e} 10} \times \log._{e} N$;

that is, *the common logarithm of any number is equal to the Napierian Logarithm of the same number, multiplied by the reciprocal of the Napierian logarithm of* 10 : *this multiplier*, $\dfrac{1}{\log._{e} 10}$, is therefore the *modulus.*

Now $\log._{e} 10 = 2{\cdot}3025850929 \ldots.;$

therefore, $M = \dfrac{1}{\log._{e} 10} = {\cdot}4342944819 \ldots.,$

which is the modulus of the common system. The common logarithms may, therefore, be calculated either by substituting this value of M in the general formula of Art. 246, or by multiplying the Napierian logarithm by the modulus. The former method is, of course, by far the more direct and expeditious : in this way we have

$$\log.\ 2 = {\cdot}86858896\left(\frac{1}{3} + \frac{1}{3^4} + \frac{1}{5 \cdot 3^5}\ \&c.\right) = {\cdot}301030.$$

$$\log.\ 3 = {\cdot}86858896\left(\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5^6} + \&c.\right) + \log.\ 2 = {\cdot}477121.$$

$$\log.\ 5 = {\cdot}86858896\left(\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5}\ \&c.\right) + \log.\ 4 = {\cdot}698970.$$

$$\log.\ 7 = {\cdot}86858896\left(\frac{1}{13} + \frac{1}{3.13^3} + \frac{1}{5 \cdot 13^5} + \&c.\right) + \log.\ 6 = {\cdot}845098.$$

250. The series here employed is well adapted to computation from its rapid convergency. When N is a large number, the convergency is so great that significant figures occur only in the remote decimal places. Thus, if N = 10000,

log. (N + 1) – log. N, or D, the difference of the two logarithms is ·0000434, correct to 7 places, beyond which, logarithms are seldom calculated. Now, log. 10000 = 4, ∴ log. 10001 = 4·0000434. For several higher numbers D will remain the same, so that the logarithms of these may be found by adding D successively. The greater the number is, and the fewer the decimal places required in the logarithm, the greater will be the series of successive numbers to which this difference will apply. These differences are placed in a column in all tables, for the purpose of deriving one logarithm from another by addition. They will be seen to diminish, and to have a wider range, towards the end of the table.

If now, d express a small excess of one number over another, then we plainly have

$$D = \log._{10}(N + d) - \log._{10}N = \log._{10} \frac{N + d}{N} = M \log._e\left(1 + \frac{d}{N}\right)$$

$$= M\left(\frac{d}{N} - \&c.\right) = \frac{M}{N}d, \text{ nearly};$$

the other terms being disregarded from their smallness. Hence D is proportional to d nearly; that is, *the difference of the logarithms is nearly proportional to the difference of the numbers, when the difference of the numbers is small.* Suppose the logarithm of 29786·93 to be required; we have from the table, log. 29786 = 4·4740122, and on the margin, 146 for the tabular difference of the mantissæ of the logs. of 29786 and 29787; we then take $\frac{93}{100}$ths of this difference, since the difference of the logarithms is proportional to the difference of the numbers; this gives 135·78, so that we add 136 to the logarithm before found, giving for the sum 4·4740258. If there be but one decimal place, we multiply by $\frac{d}{10}$; if three, by $\frac{d}{1000}$, &c. The additive numbers thus found are called "proportional parts;" they are placed upon the margin in many tables. These are, in the above example, 131 for 90, and 4 for 3.

Many other contrivances are adopted to abridge the labour of computing tables, of which some examples are given above, and others in the Exercises which follow.

APPLICATION OF LOGARITHMS.

1. Raise 9 to the ninth power.

Here $\log. 9^9 = 9 \log. 9$;

but $\log. 9 = {\cdot}9542425$,

and $\cdot9542425 \times 9 = 8{\cdot}5881825 = \log. 387420489$ $\therefore 9^9 = 387420489$.

2. Extract the 7th root of 67943.

$\text{Log}. 67943 = 4{\cdot}8321447$; $\log. \sqrt[7]{67943} = \frac{1}{7} \log. 67943 = {\cdot}6903064 = \log. 4{\cdot}901245$ $\therefore \sqrt[7]{67943} = 4{\cdot}901245$.

3. Find a fourth proportional to the 6th power of 9, the 4th power of 7, and the 5th power of 5.

Let x be the number; then

$$x = \frac{7^4 \times 5^5}{9^6} \therefore \log. x = 4 \log. 7$$

$+ 5 \log. 5 - 6 \log. 9 = \log. 14 \cdot 12$ nearly, $\therefore x = 14 \cdot 12$ nearly.

4. From the equation $2^x = 1976$ find x.

Here $x \log. 2 = \log. 1976 \therefore x = \dfrac{\log. 1976}{\log. 2} = 10 \cdot 9$ nearly.

5. Given $a^{b^x} = c$; required x.

Let $b^x = y,$

so that $a^y = c;$

then $x \log. b = \log. y,$

and $$x = \frac{\log. y}{\log. b};$$

also $$y \log. a = \log. c, \therefore y = \frac{\log. c}{\log. a},$$

hence $$x = \frac{\log. (\log. c) - \log (\log. a)}{\log. b}.$$

6. In the expression $s = \dfrac{ar^n - a}{r - 1}$ (Art. 209) required n.

Clearing and taking the logs. of both sides, we have

$$\log. a + n \log. r = \log \ \{s(r-1) + a\}$$

hence $$n = \frac{\log. \ \{s(r-1) + a\} \ - \log. a}{\log. r},$$

an expression for the number of terms in a Geometric series.

EXERCISES.

1. Given $a^x = b$; find x.
2. Given $a^{bx} = c$; find x.
3. Given $20^x = 100$; find x.
4. Given log. 2 = ·301030; find those of 25 and ·0125.
5. From the equation $(\frac{4}{4})^x = 54\frac{1}{2}$, find x.
6. Given $a^{mx} b^{nx} = c$; find x.
7. From the logarithms given in the last Art., find those of 4, 6, 8, 9, 10, 21, 32, 20, 50, and 40.
8. Calculate by the formula the logs. of 11 and 13; and hence find the logs. of 33, 39, 121, 143, and 169.
9. Given $2^{3x} . 5^{2x-1} = 4^{5x} . 3^{x+1}$, required x.
10. In the G. P. 729, 486, 324, &c., find the number of that term which is 64; and also the number of terms whereof the sum is $2130\frac{1}{9}$.

11. Given $x^y = y^x$ and $x^p = y^q$; find x and y.
12. From the log. of 2, deduce those of 5, 16, ·016, 6·25 and $3\frac{1}{8}$.
13. Given log. 80 = 1·9030900, log. 81 = 1·9084850, log. 82 = 1·9138139, log. 84 = 1·9242793; required the log. of 83 by interpolation.
14. Given log. 101 = 2·0043214, log. 102 = 2·0086002, log. 104 = 2·0170333, log. 105 = 2·0211893; find the log. of 103 by interpolation.

XXV.—PERMUTATIONS AND COMBINATIONS.

251. The various arrangements which may be formed of any things, taking the whole or a certain number of them together, are called their *permutations.*

Thus the permutations of the letters *a*, *b*, *c*, taken two at a time, or two and two together, are *ab*, *ba*, *ac*, *ca*, *bc*, *cb*; taken three together, the permutations are *abc*, *acb*, *bac*, *bca*; and it is obvious that in either case no more than these are possible.

It is to be observed here that the juxtaposition of the letters has no reference to the formation of products, but indicates merely an order of succession.

252. The different groups that can be formed of any things, without regard to the order in which the things are placed, are called their *combinations.*

Thus the combinations of the letters *a*, *b*, *c*, taken two at a time, are *ab*, *ac*, *bc*. The number of combinations in this case is plainly *half* the number of permutations, since *ab*, *ba*, &c., though different permutations, form but one combination. If we take the same letters three together, there is but *one* combination *abc*, while there are *six* permutations.

253. It is obvious that the number of permutations of a, b, c, d, &c. . . . n things taken singly or separately, is simply n. And if one of these, a, be omitted, the number of permutations of the $(n-1)$ remaining, taken singly, will be $(n-1)$; and if a be placed before each of these there will be $(n-1)$ permutations, in which a stands first, namely, *ab, ac, ad*, &c. Similarly there will be $(n-1)$ permutations, *ba, bc, bd*, &c., in which b stands first; $(n-1)$ in which c stands first, and so on. There will thus be altogether n times $(n-1)$ permutations of the form *ab, ac, ad*, &c., *ba, bc, bd*, &c.; that is, the number of permutations of n things taken two together is

$$n(n-1) \quad . \quad . \quad . \quad . \quad (1)$$

254. Again, abstracting any one of the n things, the remaining $(n-1)$ things b, c, d, &c. may be shewn, as above, to form $(n-1)(n-2)$ permutations of two and two; and prefixing a to each of these permutations, they will become permutations of three and three, in which a stands first. Similarly there will be $(n-1)(n-2)$ permutations in which b stands first, and so on. And there will thus be n times $(n-1)(n-2)$ permutations in all; that is, the number of permutations of n things, taken three and three, is

$$n(n-1)(n-2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the same way it may be shewn that the number of permutations of n things, taken four and four together, will be

$$n(n-1)(n-2)(n-3) \quad . \quad . \quad . \quad . \quad (3)$$

By a similar process of reasoning, it may be inferred that the permutations of n things, taken r together, is $n(n-1)$ $(n-2)$ . . . . $[n-(r-1)]$, or expressing the number of these by P_r,

$$P_r = n(n-1)(n-2) \ldots (n-r+1) \quad . \quad . \quad (4)$$

255. If the n quantities be taken all together, then $n = r$ and the number of permutations therefore is $n(n-1)(n-2)$ . . . . . $[n-(n-3)][n-(n-2)][n-(n-1)]$.

In this the last term becomes $n - n + 1 = 1$; the last but one $n - n + 2 = 2$, &c. So that the number of permutations is

$$P_n = n(n-1)(n-2)(n-3) \ldots 3 . 2 . 1 \quad . \quad (5)$$

or inverting the order of the factors

$$1 . 2 . 3 . 4 \ldots . (n-3)(n-2)(n-1)n$$

or

$$1 . 2 . 3 . 4 \ldots\ldots\ldots . n \quad . \quad . \quad . \quad . \quad (5')$$

Thus the number of permutations of four things taken all together is 4 . 3 . 2 . 1 or 1 . 2 . 3 . 4 = 24; of five things 1 . 2 . 3 . 4 . 5 = 120, n being in the one case 4, and in the other 5.

256. If out of the n things p be alike, then these p things taken all together will form but one permutation instead of $1 . 2 . 3 \ldots . p$ permutations, (5); so that the whole number will be diminished $1 . 2 . 3 \ldots . p$ times. Hence, calling P the number of permutations of n things, p of which are alike, we shall have

$$P = \frac{1 . 2 . 3 . 4 \ldots n}{1 . 2 . 3 . 4 \ldots p} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If again there be also q of them the same, though different from the former, P will be diminished $1 . 2 . 3 \ldots q$ times; and hence in this case

$$P = \frac{1 . 2 . 3 . 4 \ldots n}{(1 . 2 . 3 \ldots p)(1 . 2 . 3 \ldots q)} \quad . \quad . \quad (7)$$

And, similarly, if there be r others of the things which are the same; and if for the sake of brevity the continued product $n(n-1)(n-2) \ldots 3 . 2 . 1$ or $1 . 2 . 3 . \, . n$ be denoted by $\underline{|n}$ we shall have

$$P = \frac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

257. We have seen already (Art. 252), that if n things are taken two and two, the number of combinations is half the number of permutations; and if they are taken three and three, each combination affords 6 or 2 . 3 permutations; that is, calling C_2 and C_3 the number of combinations of n things, taken two and two, and three and three, we have

$$C_2 = \frac{n(n-1)}{1 . 2}; \qquad C_3 = \frac{n(n-1)(n-2)}{1 . 2 . 3}.$$

Generally, since every combination of r things admits of $1 . 2 . 3 \ldots r$ permutations, we shall have by Art. 254,

$$C_r = \frac{n(n-1)(n-2) \ldots (n-r+1)}{\underline{|r}} \quad . \quad . \quad . \quad (9)$$

If now, both terms of this expression be multiplied by $\underline{|n-r}$, it will become

$$C_r = \frac{n(n-1) \ldots (n-r+1)(n-r) \ldots 3 . 2 . 1}{\underline{|r}\,\underline{|n-r}},$$

or

$$C_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

258. If in (9) last Art. $n-r$ be put for r, we shall have, after reduction in the numerator,

$$C_{n-r} = \frac{n(n-1)(n-2) \ldots (r+1)}{\underline{|n-r}};$$

and if both terms of the right member be multiplied by $\underline{|r}$ the expression becomes

$$C_{n-r} = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Hence by last Art. $C_r = C_{n-r}$; that is, *the number of combinations of* n *things taken,* r *and* r, *together, is the same as the number of them taken* n − r *together.* These combinations are called complementary.

Thus, to find the number of combinations of 20 things taken 16 together; by (9) we have

$$n = 20, r = 16 \therefore n - r + 1 = 5, C_r = \frac{20 . 19 . 18 . \dots 5}{1 . 2 . 3 . 4 . \dots 16} = 4845.$$

But the operation will be shorter by taking them $n - r$; that is, $20 - 16 = 4$ together. We thus get

$$C_4 = \frac{n(n-1)(n-2)(n-3)}{1 . 2 . 3 . 4} = \frac{20 . 19 . 18 . 17}{1 . 2 . 3 . 4}$$
$$= 5 . 19 . 3 . 17 = 4845.$$

In the other expression the factors cancel so as to give the same as these; but the latter process is the readier; only however, when $r > \frac{1}{2}n$.*

259. Supplying the terms before the last, (9) becomes

$$C_r = \frac{n(n-1) \dots (n-r+2)(n-r+1)}{1 . 2 . 3 \dots (r-1)r}.$$

Also by substitution of $r - 1$ for r in the same formula, we have

$$C_{r-1} = \frac{n(n-1) \dots (n-r+2)}{1 . 2 . 3 (r-1)}$$

Hence $$C_r = \frac{n-r+1}{r} . C_{r-1}$$

Since, then $C_r > C_{r-1}$, the number of combinations will increase as r increases, so long as $\frac{n-r+1}{2} > 1$; that is, until it is equal to 1, or less than 1. If the value of r be such that this factor is unity; that is, that $n - r + 1 = r$, or $r = \frac{n+1}{2}$ (n being odd, because r must always be an integer), then this value of r will give the greatest number of combinations. Now $\frac{n-1}{2}$ expresses the complementary combinations, since $n - \frac{n+1}{2} = \frac{n-1}{2}$; so that *for* n *odd, the number of combinations is the greatest when taken* $\frac{1}{2}(n+1)$, *or* $\frac{1}{2}(n-1)$ *together.* Let, now, n be even, and put $n = 2m$; then $\frac{2m+1}{r} - 1 > 1$, until $r = m$, and also for this value of r; but if $r = m + 1$, then $\frac{2m+1}{r} - 1 < 1$. *Hence the value* m *for* r, *that is,* $\frac{n}{2}$ *for* r, *gives the greatest number of combinations of* n *things when* n *is even. See* the Coefficients of the Powers, top of p. 92.

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* For further illustrations of the nature of these complementary combinations, see the coefficients of the powers at the top of p. 92.

260. The number of combinations of n sets of things, containing respectively, p, q, r, &c. things, one being taken out of each set for each combination, will plainly be, for one from each of two sets, pq; for a third set of r things, each of them combined with these, pqr, and so on, the combinations being expressed by the continued product. If we have $p = q = r$, &c., or the same number in each set, then the number of combinations is expressed by p^n.

261. The number of combinations of two sets of things containing p and q things respectively, m being taken from one set, and n from the other for each combination, will plainly be had by taking the product of two expressions, the same as formula (9); that is, the number of combinations will be

$$\frac{p(p-1)\ .\ .\ .\ .\ (p-m+1)}{\underline{|m}} \times \frac{q(q-1)\ .\ .\ .\ .\ (q-n+1)}{\underline{|n}}.$$

EXERCISES.

Find the number of permutations,

1. Of 12 things taken 2 together, 3 together, and 4 together.

2. Of 7 things taken 3 together, 4 together, 5 together, and 6 together.

3. Of 8 things taken 5 together and all together.

4. Of the letters of the following words taken all together in each case; Profundity; Algebra; Mississippi; Caraccas; Ecclesiastical.

5. Of the letters of the product $a^3b^5c^2$, written fully out.

Find the number of combinations,

6. Of 24 things taken 2 together, 3 together, and 4 together.

7. Of 24 things taken 22 together, 21 together, and 20 together.

8. Of 24 things taken 11 together, 12 together, and 13 together; and shew how this Exercise, and also Ex. 6 and 7, illustrate Arts. 258 and 259.

9. Of 9 things taken 2 and 2, 3 and 3, &c. . . . 8 and 8 together; arrange the results in a series, and shew how this illustrates Art. 258.

10. Of 52 things taken 13 together; that is, how many different hands a person may hold at whist.

11. Of 10 things taken 3 together, and 7 together.

12. If the number of permutations of n things taken 3 together be to the number of combinations of n things taken 4 together, as 6 : 1; required n.

13. A party of 25 men supply 4 sentinels each night. On how many different nights can a different selection be made; and on how many of these will any particular man be engaged?

14. If there are 20 pears at 3 a-penny, how many different selections can be made in buying six pennyworth; and in how many of these will a particular pear occur?

15. For how many days can six persons be placed in different positions about a table at dinner?

16. Out of 12 consonants and 5 vowels, how many different collections of letters can be formed with 6 consonants and 3 vowels?

17. Out of 12 democrats and 16 republicans, how many different committees can be formed, each consisting of 3 democrats and 4 republicans?

18. Find the number of different triangles into which a polygon of n sides may be divided by joining the angles.

19. A party consists of 10 Englishmen, 8 Frenchmen, 7 Germans, and 5 Italians; required the number of ways in which they can be taken, so as to have always one, and no more of each nation. Required the same, supposing that there were 10 of each nation.

20. Of 8 things, how many must be taken together, that the number of combinations may be the greatest possible?

21. How many different sums may be formed with a sovereign, a half-sovereign, a crown, a half-crown, a shilling, and a sixpence? *See* Ex. 5, p. 92.

22. Find the number of words of 6 letters, each of which may be formed out of the first 10 letters of the English alphabet, each word being supposed to contain 2 vowels.

23. Three cards being dealt to each person of a party at play, any one can have 425 times as many hands as there are cards in the pack. Required the number of cards.

24. At an election, where every voter may vote for any number of candidates not greater than the number to be elected, there are 4 candidates and 3 members to be chosen; in how many ways may a man vote?

XXVI. BINOMIAL THEOREM.

262. The truth of this theorem, by far the most important in the whole range of Algebra, has been already (Art. 91) inferred by analogy from the mode in which the powers of binomials are developed. A general proof will now be given, derived by induction and the theory of combinations, from the forms which the products of binomial factors assume in successive multiplication. The theorem will next be demonstrated by the method which has so many resources, and is so extensively applicable in every part of analysis, that of indeterminate coefficients.

263. By successive multiplication the following products are easily formed;

$$(x+a)(x+b)=x^2+(a+b)x+ab$$

$$(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$$

$$\begin{aligned}(x+a)(x+b)(x+c)(x+d)=x^4&+(a+b+c+d)x^3\\&+(ab+ac+bc+ad+bd+cd)x^2\\&+(abc+acd+bcd+abd)x+abcd.\end{aligned}$$

From these the law of continuation is manifest; the indices of x decrease by a unit in each term from the first, in which the index is the same as the number of factors; the coefficient of this term is unity; that of the second is the sum of all the second terms of the binomials; the third coefficient is the sum of these second terms, taken two and two together; the fourth, that of the same quantities, taken three and three, and so on; and the last term is the product of these same quantities, taken all together. Since, then, in the *third* term, these quantities are taken *two* and *two* together; in the *fourth*, *three* and *three*, and so on, by induction, it may be safely inferred that in the $(n+1)$th term they would be taken n and n together.

Let us suppose, then, that this law holds for the product n factors multiplied together continually; and denoting, for the sake of brevity, the coefficients after the first term by A, B, C, &c., we shall have

$$(x+a)(x+b)(x+c)\ldots(x+k)=x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\ldots K$$

in which $A=a+b+c\ldots+k \qquad B=ab+ac+bc+\ldots$

$$C=abc+acd+\ldots \qquad K=abcd\ldots k.$$

Supposing, again, that there are $(n+1)$ factors, we obtain

$$(x+a)(x+b)(x+c)\ldots(x+k)(x+l)=x^{n+1}+(A+l)x^n$$
$$+(B+lA)x^{n-1}+\ldots+Kl$$

in which $A+l=a+b+c+\ldots+k+l$,

$B+lA=ab+ac+bc+\ldots+al+bl+\ldots+kl$, &c., &c.,

and $Kl=abcd\ldots Kl$.

The law is therefore proved to hold for $(n+1)$ factors, if it holds for n factors. Now, it is seen above to hold for 4, therefore it holds for $(n+1)$ or 5; and if for 5 then for 6, and so on. We are thus entitled to infer that it holds for any number.

Let now $a=b=c$, &c.; then $(x+a)(x+a)\ldots$ to n factors $=(x+a)^n$. Also $A=a+a+a\ldots$ to n terms $=na$; $B=a^2+a^2\ldots$ to as many terms as there are combinations of n things taken *two* together $=\frac{n(n-1)}{1.2}a^2$; $C=a^3+a^3\ldots$ to as many terms as there are combinations of n things taken *three* together $=\frac{n(n-1)(n-2)}{1.2.3}a^3$.

Similarly $D=\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}a^4$.

Also $K=a.a.a\ldots$ to n factors $=a^n$. The general formula therefore is

$$(x+a)^n=x^n+nax^{n-1}+\frac{n(n-1)}{1.2}a^2x^{n-2}+\frac{n(n-1)(n-2)}{1.2.3}a^3x^{n-3}$$
$$+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}a^4x^{n-4}+\&c.\ldots+a^n \quad . \quad . \quad (1)$$

Since x and a in this investigation may have any value; for x put a and for a put x; the formula then becomes

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3 \;.\;.\;.\;.\;.\; x^n . \quad . \quad (1')$$

If $a = 1$, this formula becomes

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 \text{ \&c.} \ldots + x^n \quad (2)$$

264. If the second term of the binomial be negative, that is, if for x we put $-x$, then the odd powers will be negative; but these occur in the even terms of the expansion, so that these even terms are negative. *See* Art. 92. The formula then becomes

$$(a-x)^n = a^n - na^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 - \text{\&c.}$$

and similarly

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{1.2}x^2 - \frac{n(n-1)(n-2)}{1.2.3}x^3 + \text{\&c.}$$

265. This proof of the binomial theorem from induction, proceeds upon the supposition that n is a positive integer. We shall now, by help of the method of indeterminate coefficients, give a general proof, independent of induction, for any index, integral or fractional, positive or negative.

Put $(a+x)^n$ under the form $a^n\left(1+\frac{x}{a}\right)^n$; and if we make $\frac{x}{a} = y$, we have to find the development of $(1+y)^n$; this being found and multiplied by a^n, we shall have the complete development required.

Let now the exponent be any positive fraction $\frac{p}{q}$, p and q

being positive integers; then (Art. 159), we may assume

$$(1+y)^{\frac{p}{q}} = 1 + Ay + By^2 + Cy^3 + Dy^4 + \&c. \quad . \quad . \quad (1)$$

the first term being 1, because, when $y = 0$, the series is reduced to $1 = 1$.

Similarly, we may assume

$$(1+z)^{\frac{p}{q}} = 1 + Ay + By^2 + Cy^3 + Dy^4 + \&c. \quad . \quad . \quad (2)$$

in which the coefficients have plainly the same value as in (1) since they are independent of any values of y and z, and so are true for any, and therefore for the value $y = z$.

If now we put $(1+y)^{\frac{1}{q}} = u$, and $(1+z)^{\frac{1}{q}} = v$, then $1 + y = u^q$, and $1 + z = v^q$, and $\therefore$ $y - z = u^q - v^q$, $(1+y)^{\frac{p}{q}} = u^p$, and $(1+z)^{\frac{p}{q}} = v^p$. Bearing this in mind, and subtracting (2) from (1) we shall obtain

$$u^p - v^p = A(y-z) + B(y^2 - z^2) + C(y^3 - z^3) + \&c. \quad . \quad . \quad (3)$$

Dividing the left member of this equation by $u^q - v^q$, and the right by its equal, $y - z$, we have

$$\frac{u^p - v^p}{u^q - v^q} = \frac{A(y-z) + B(y^2 - z^2) + C(y^3 - z^3)}{y - z} + \&c. \quad . \quad (4)$$

In this equation the right member is plainly reducible by $y - z$; and both terms of the left member are divisible (Art. 106) by $u - v$, so that this factor may be suppressed. Performing the division (4) becomes

$$\frac{u^{p-1} + vu^{p-2} + v^2u^{p-3} \ldots + v^{p-1}}{u^{q-1} + vu^{q-2} + v^2u^{q-3} \ldots + v^{q-1}}$$

$$= A + B(y+z) + C(y^2 + yz + z^2) + D(y^3 + y^2z + yz^2 + z^3), \&c. \ (5)$$

Now, as this equation is perfectly general, it is true for the value $y = z$; whence $1 + y = 1 + z$, and $u = v$: hence each term of the numerator of the left member is u^{p-1}, and each of the denominator u^{q-1}; so that the left member becomes $\frac{p}{q} \cdot \frac{u^{p-1}}{u^{q-1}}$; or by multiplying both terms by u, $\frac{p}{q} \cdot \frac{u^p}{u^q}$;

also the terms within the vincula on the right side become $2y$, $3y^2$, &c. Equation (5) thus becomes

$$\frac{p}{q} \cdot \frac{u^p}{u^q} = A + 2By + 3Cy^2 + 4Dy^3, \text{ \&c.} \quad . \quad . \quad . \quad . \quad (6)$$

Now $u^p = (1+y)^{\frac{p}{q}}$, and $u^q = 1 + y$; hence, supplying from (1) the value of $(1+y)^{\frac{p}{q}}$, we have

$$\frac{p}{q} \cdot \frac{1 + Ay + By^2 + Cy^3 + \text{\&c.}}{1 + y}$$
$$= A + 2By + 3Cy^2 + 4Dy^3 + 5Ey^4 + \text{\&c.} \quad . \quad . \quad . \quad . \quad (7)$$

Multiplying both members by $1 + y$, and arranging the terms of the product on the right side according to the powers of y, we obtain for the final equation

$$\frac{p}{q}(1 + Ay + By^2 + Cy^3 \text{ \&c.})$$
$$= A + (2B + A)y + (3C + 2B)y^2 + (4D + 3C)y^3 + \text{\&c.} \quad (8)$$

From this we can easily deduce, as in Art. 159, the following values of the coefficients—

$$A = \frac{p}{q},\ B = \frac{1}{2}\,\frac{p}{q}\left(\frac{p}{q} - 1\right),\ C = \frac{1}{2}\cdot\frac{1}{3}\,\frac{p}{q}\left(\frac{p}{q} - 1\right)\left(\frac{p}{q} - 2\right)$$
$$D = \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\cdot\frac{p}{q}\left(\frac{p}{q} - 1\right)\left(\frac{p}{q} - 2\right)\left(\frac{p}{q} - 3\right), \text{ \&c.}$$

Let M be the rth term from the beginning, and N the $(r + 1)$th; then $\frac{p}{q}M = Nr + M(r - 1)$;

whence $$N = \frac{M}{r}\left(\frac{p}{q} - r + 1\right).$$

We have thus, since the powers of y are known, the complete development of the binomial $(1 + y)^{\frac{p}{q}}$ *

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* This proof being independent of induction, and of a previous acquaintance with the form of the development when the index is a positive integer, is much more satisfactory than the proof usually given, called Euler's proof, and than that which presupposes a knowledge of the extraction of roots. It is, besides, very easy, and founded upon principles recognised in elementary algebra.

266. If the index be a negative fraction, one part of the foregoing investigation will require to be slightly modified, in order to obtain the development in this case. Equation (3) will become

$$u^{-p} - v^{-p} = A(y - z) + B(y^2 - z^2) + C(y^3 - z^3), \text{ \&c.}$$

$$\text{Now } u^{-p} - v^{-p} = \frac{1}{u^p} - \frac{1}{v^p} = \frac{v^p - u^p}{u^p v^p} = - \frac{u^p - v^p}{u^p v^p}.$$

Then in dividing, as before, the first member by $u^q - v^q$, and the second by $y - z$, its equal, we find

$$- \frac{1}{u^p v^p} \cdot \frac{u^p - v^p}{u^q - v^q} = \frac{A(y - z) + B(y^2 - z^2) \text{ \&c.}}{y - z}.$$

Then suppressing, as before, the factors $u - v$, and $y - z$, we obtain

$$- \frac{1}{u^p v^p} \cdot \frac{u^{p-1} + vu^{p-2} + \ldots v^{p-1}}{u^{q-1} + vu^{q-2} \ldots v^{q-1}} = A + B(y + z) + C(y^2 - z^2)$$

&c.

Putting $y = z$, whence $u = v$, the above expression becomes

$$- \frac{1}{u^{2p}} \cdot \frac{p}{q} \cdot \frac{u^{p-1}}{u^{q-1}} = - \frac{p}{q} \cdot \frac{u^{-p}}{u^q} = A + 2By + 3Cy^2 + \text{\&c.}$$

Now this form is precisely the same as equation (6), except that the index p is in this case negative. The successive coefficients will therefore be found in this case by merely changing the sign of p in the values already given. This gives us the development in the case of a negative fraction.

267. If in the investigation of Art. 265, we suppose $q = 1$, we obtain the development for a positive integral index, the same as formula (2), Art. 263; and if in that of Art. 266, $q = 1$, then the index is $-p$; and those coefficients of the development will be negative into which this quantity enters *an odd number* of times as factor; that is, those in

the *even terms;* for, $-p(-p-1)$ may be replaced by $p(p-1)$, and $-p(-p-1)(-p-2)$ by $-p(p-1)(p-2)$, &c. Hence, universally, whatever value the exponent n may have, whether of the form $\pm \frac{p}{q}$, or $\pm p$, the development is of the same form, and we have only to substitute $\frac{x}{a}$ for y, and multiply by a^n, in order to obtain the general form already deduced.

268. If either term of the binomial be negative, and the index be also negative, then manifestly all the terms will be positive (Art. 264); so that the development of $(a-x)^{-n}$ is the same as that of $(a+x)^n$.

269. To find the rth term we have only to observe that

$$\frac{n(n-1)}{1\,.\,2} = \frac{n(n-3+2)}{1\,.\,(3-1)}, \quad \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3} = \frac{n(n-1)(n-4+2)}{1\,.\,2\,.\,(4-1)}$$

&c.; and that $a^{n-2}x^2 = a^{n-3+1}x^{3-1}$, $a^{n-3}x^3 = a^{n-4+1}x^{4-1}$, &c., and we shall at once see that the rth term will be

$$\pm \frac{n(n-1)(n-2)\,.\,.\,.\,(n-r+2)}{1\,.\,2\,.\,3\,.\,(r-1)} a^{n-r+1}x^{r-1}.$$

This is the *general term of the series*, because any term after the first may be had from it, by putting 1, 2, 3, &c. for r.

270. Since the coefficient of the rth term is the same as $\frac{n(n-1)\,.\,.\,.\,\{n-(r-1)+1\}}{1\,.\,2\,.\,3\,.\,(r-1)}$, which expresses the number of combinations of n things taken $(r-1)$ together (Art. 259), it follows that this coefficient, that is, every coefficient, is necessarily a whole number, n being a positive integer.

271. It is obvious that the $(r+1)$th term $=$ rth $\times$ $\frac{n-r+1}{r}\cdot\frac{x}{a}$. Hence the rth term will be greatest when $\frac{x}{a}\cdot\frac{n-r+1}{r}=1$, or <1; that is, when $x(n-r+1)=ra$ or $<ra$; that is, when $x(n+1)=$ or $<r(x+a)$, or when $r=$ or $>\frac{(n+1)x}{x+a}$; $\therefore$ r must be the whole number equal to, or next greater than $\frac{(n+1)x}{x+a}$, according as the value of this latter is integral or fractional. If it be an integer, the rth and $(r+1)$th terms are plainly equal, and each greater than any other term; but if it be *not* an integer, then the greatest term is that denoted by the number expressing the integral part of the quotient. Thus, if the greatest term in the expansion of $(3+5x)^8$ be required; when $x=\frac{1}{2}$, we have $\frac{(n+1)x}{a+x}=(8+1)\cdot\frac{5}{2}\cdot\frac{1}{3+\frac{5}{2}}=\frac{45}{11}=4\frac{1}{11}$. The *first* whole number greater than $4\frac{1}{11}$ is 5; $\therefore$ the 5th is the greatest term; from this term the terms become less and less, or the series begins to converge. The formula is applied to binomials of the form $(1\pm x)^n$, which lead to series by their expansion; x being plainly less than unity, else the terms might increase without limit.

272. In the expansion of $(a+x)^n$, n being a positive integer, there are $(n+1)$ terms, and no more; for the coefficient of this term is

$$\frac{n(n-1)\ldots\{n-(n+1)+2\}}{1.2.3\ldots\{(n+1)-1\}}=\frac{n(n-1)\ldots 2.1}{1.2\ldots(n-1)n}=1,$$

while the last factor of the numerator of the coefficient of the $(n+2)$th term is $n-(n+2)+2=0$, so that the entire term vanishes, as would also every subsequent term from the presence of this factor. *See* Art. 91, 2d.

273. In formula (1′), Art. 263, true for all values of the terms, let $a = 1$, $x = 1$, then $(1+1)^n = 2^n = 1 + n + \frac{n(n-1)}{1\,.\,2}$ &c.; that is, *the sum of the coefficients equals the* n*th power* of 2. Examples of this remarkable relation will be had from the expansions given in p. 92.

274. In the same formula let $a = 1$, $x = -1$, then $(1-1)^n$ $= 0^n = 0 = 1 - n + \frac{n(n-1)}{1\,.\,2} - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}$, &c. $= \left\{1 + \frac{n(n-1)}{1\,.\,2} + \text{\&c.}\right\} - \left\{n + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3} + \text{\&c.}\right\}$ = sum of coefficients in even terms – sum in odd terms.

Hence, since these sums are equal, *each of them* must equal half the sum of all the coefficients; that is, the sum of the coefficients of the even terms, *or* the sum of those of the odd terms $= \frac{2^n}{2} = 2^{n-1}$.

275. The greatest coefficient in the expanded binomial can at once be had from the formula $\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$ (Art. 257), by substituting $\frac{n}{2}$ for r when n is even, and $\frac{n-1}{2}$ or $\frac{n+1}{2}$ for r when n is odd; the first result giving one middle term, and the second two, equal to one another. The greatest coefficient for n even, is at the distance $\frac{n}{2} + 1$ from the beginning, with $\frac{n}{2}$ terms on each side of it, since the number of factors $\frac{n}{1}, \frac{n-1}{2}, \frac{n-2}{3}$, &c. is less by unity than the number denoting the place of a term; that is, the combination of 2 and 2 occurs in the third place, of 3 and 3 in the fourth,

or generally the coefficient of the $(r+1)$th term is the number of combinations of n things taken r at a time, &c.; the greatest for n odd are in the places denoted by $\frac{n+1}{2}$ and $\frac{n+1}{2}+1=\frac{n+3}{2}$, while there are $\frac{n-1}{2}$ terms on either side. The coefficients being computed as far as these middle terms, those that follow will be the same, by the principle of complementary combinations, as those which precede these terms, but in a reverse order. Or the same may be proved generally, as follows:

The coefficient of the rth term from the beginning (Art. 269), becomes, when both terms are multiplied by $1 \, . \, 2 \, . \, 3 \, \ldots \, \{n-(r-1)\}$, or $\underline{|n-r+1}$,

$$\frac{\underline{|n}}{\underline{|r-1}\,\underline{|n-r+1}}$$

Now the rth term from the end, having $(r-1)$ terms after it, is the $(n+1)$th $-$ $(r-1)$th from the beginning, that is, the $(n-r+2)$th from the beginning. The coefficient of this term is

$$\frac{n(n-1)\ldots\ \{n-(n-r+2)+2\}}{\underline{|n-r+1}} \text{ or } \frac{n(n-1)\ldots r}{\underline{|n-r+1}};$$

this also is equal to

$$\frac{\underline{|n}}{\underline{|r-1}\,\underline{|n-r+1}}$$

so that the coefficient of the rth term from the beginning equals that of the rth term from the end.

It is only necessary to give a very few additional Exercises in this place, as the principles now laid down may be illustrated from Sections VIII. and IX.

EXERCISES.

1. Expand $\left(1+\frac{x}{2}\right)^6$; $\left(1-\frac{x}{2}\right)^{-2}$; $(a+2b-c)^3$, and $\left(\frac{a+x}{a-x}\right)^n$.

2. Find the greatest terms in the series $(1+5x)^{\frac{3}{2}}$, and $(1+5x)^{-\frac{3}{2}}$, when $x=\frac{1}{2}$; and in $(1+5x)^8$, when $x=\frac{1}{6}$.

3. At what term does the series for $(1+9x)^{\frac{4}{3}}$ begin to converge when $x=\frac{1}{10}$?

4. Find the middle term in the expansion of $(a^2+x^3)^{12}$; and the two middle terms in that of $(a+x)^{13}$.

5. Find the *r*th term in the expansion of $(a+x)^{\frac{1}{2}}$; and the 7th in that of $(a^3+3ax)^9$.

6. Find the middle term in the expansion of $(1+x)^{2n}$; and shew that the coefficient of the $(r+1)$th term of $(1+x)^{n+1}$ equals the sum of the *r*th and $(r+1)$th coefficients of $(1+x)^n$.

XXVII.—PROBABILITY.

276. There is a large class of events which happen according to laws of which we are ignorant, or which are of such a nature that *we can see no reason* why they should happen in one way rather than in another. From previous experience, or otherwise, we may have a *greater* or *less degree* of expectation of the event happening in one such way, rather than another; but, unless we can reduce this degree of expectation to a numerical measure or estimate, we cannot bring the subject of the occurrence of such events within the domain of mathematics. Now, the relation of greater and less enters into our idea of probability; we speak of one

event as more probable, much more probable than another; but this relation is one which belongs to magnitude generally; and magnitude is the object of mathematics.

If we can speak of one event as more probable than another, then are there greater and lesser degrees of probability; one probability may be said to be *twice, three times*, &c., as great as another; they may be compared, in respect of magnitude, and ratios instituted between them. These magnitudes then, and the measures of their ratios, may be expressed by letters and common numbers, as in the case of magnitudes ordinarily considered in mathematics, such as lines, and angles, and their ratios; time, space, velocity, force, &c. It is hence obvious, that we must have here as in other cases, a unit of reference by which to estimate the ratios in question. Certainty, represented by unity, has been adopted for this purpose. The probability 1 for an event will thus denote certainty in its favour; certainty against an event will be denoted by the probability 0 in its favour; while intermediate degrees of probability will come to be represented by fractions ranging between 0 and 1. From these considerations as a groundwork, the whole mathematical theory is deduced. It is now the subject of separate treatises, and no more can be given here than a few elementary propositions. The following example will illustrate the foregoing remarks.

Suppose 5 white and 7 black balls placed at random in a box, and that 1 is to be drawn by a person who does not see the balls; it is obvious that a black ball is more likely to be drawn than a white one, and *that* in the ratio of 7 to 5. since there is no determining cause whatever, except the number of black balls; the majority being black, a black is more likely to come up than a white, in the ratio of the numbers 7 and 5. That is, $\frac{7}{12}$ is the probability of drawing a black ball, and $\frac{5}{12}$ that of drawing a white one. In this case, the *odds* are said to be $1\frac{2}{3}$ to 1 *in favour* of a black ball, or *against* a white ball.

Suppose, again, that there are ten white balls, marked 1, 2, 3, &c., the probability that 1 will be drawn is $\frac{1}{10}$; that either 1 or 2 will be drawn $\frac{2}{10}$; either 1, 2, or 3 $\frac{3}{10}$, &c., and $\frac{10}{10} = 1$, the certainty that some one ball will come up.

Hence, generally, if an event may happen in n ways, each equally possible, then, representing certainty by 1, the probability that the event will happen in any given way, is represented by $\frac{1}{n}$, in either of 2 ways by $\frac{2}{n}$, and of its happening in any one of m specified ways, by $\frac{1}{n} + \frac{1}{n} + \frac{1}{n}$, &c., to m terms $= m \cdot \frac{1}{n} = \frac{m}{n}$.*

277. If an event may happen in a ways, and fail in b ways, then the probability of its happening is to the probability of its failing, as a to b; hence, the probability of its happening, is to the sum of the probabilities of its happening and failing, that is, to certainty, as a to $a + b$; so that certainty being unity, $\frac{a}{a+b}$ denotes the probability of the event happening. The probability of its failing is $1 - \frac{a}{a+b} = \frac{b}{a+b}$. Here the odds are a to b in favour of the event, or b to a against it.

Similarly, if there are three events whose probabilities are proportional to the quantities a, b, c, then the probability of each happening, is given by the respective expressions,

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* Chance is used by many writers as synonymous with probability; while others, with more propriety, limit the meaning to the lowest degree of probability.

$$\frac{a}{a+b+c},\quad \frac{b}{a+b+c},\quad \frac{c}{a+b+c},\ \text{and so on.}$$

278. The number of combinations of n things, taken 2 and 2, being $\frac{1}{2}n(n-1)$, the probability of drawing any given pair, A, B, of n balls, will be $1 \div \frac{1}{2}n(n-1) = \frac{2}{n(n-1)}$. Thus suppose there are five white and seven black balls, the number of pairs that can be formed with 12 things is $\frac{1}{2} . 12 . 11 = 66$; and with 5, $\frac{1}{2} . 5 . 4 = 10$; hence, the probability of drawing two white balls is $\frac{10}{66}$; similarly, that of drawing two black is $\frac{21}{66}$. Since each white ball may be taken with each black, the number of pairs thus formed will be $7 \times 5 = 35$; and the probability of drawing a white and a black is $\frac{35}{66}$.

279. Let there be two independent events whose respective probabilities are known, and let a, a', be the number of ways in which they may happen, and b, b', those in which they may fail, all these ways being equally likely; then the probability that they will both happen is denoted by

$$\frac{a}{a+b} \times \frac{a'}{a'+b'}.$$

For each case out of the $a + b$ may be combined with each out of $a' + b'$, forming $(a + b)(a' + b')$ combinations, in which the events are equally likely to occur. In aa' of these cases both events happen; in bb' of them they fail; in ab' of them the first happens, and the second fails, and in $a'b$ of them the first fails and the second happens. Hence the probability that both happen, is

$$\frac{aa'}{(a+b)(a'+b')};$$

that both fail, is $\dfrac{bb'}{(a+b)(a'+b')}$;

first happens and second fails $\dfrac{ab'}{(a+b)(a'+b')}$;

first fails and second happens $\dfrac{a'b}{(a+b)(a'+b')}$.

Thus if m and n be the probabilities of two independent events; mn is the probability of both happening. And similarly, if there are three independent events, the probability that they all happen is mnp, and that they fail, $(1-m)(1-n)(1-p)$; that one happens and the rest fail, $m(1-n)(1-p)$, &c.

Thus, the probability of throwing an ace, then a deuce, with one die, is in the first trial $\frac{1}{6}$, and in the second $\frac{1}{6}$; $\therefore$ the probability of both happening is $\frac{1}{36}$; the probability of throwing an ace, once at least, with one die in two trials, is $\frac{11}{36}$; for the chance of failing each time is $\frac{5}{6}$, and of failing twice together, $\frac{25}{36}$; and $1-\frac{25}{36}=\frac{11}{36}$, the chance of not failing both times. Again, suppose that there are four white balls, five black, and six red; required the probability of drawing two red balls, 1st, when the ball first drawn is replaced, and 2nd, when it is not replaced, before the second trial. Here, 1st, the chance of drawing a red ball at first is $\frac{6}{15}$, at the second trial also $\frac{6}{15}$, $\therefore$ the chance of drawing two is $(\frac{6}{15})^2$. But 2nd, the chances now are $\frac{6}{15}$, and $\frac{5}{14}$, and the chance of drawing two red $\frac{6}{15}\times\frac{5}{14}=\frac{1}{7}$. This latter result is the same as would be found by Art. 278; for $\frac{1}{2}\,.\,15\,.\,14=105$, and $\frac{1}{2}\,.\,6\,.\,5=15$, and $\frac{15}{105}=\frac{1}{7}$.

In how many trials may a person undertake, for an even wager, to throw an ace with a single die; that is, in how many trials will the probability amount to $\frac{1}{2}$? Let x be the number of trials, then the chance of failing x times is $(\frac{5}{6})^x$; then, by the question, this is equal to the chance of happening; that is $(\frac{5}{6})^x = \frac{1}{2}$; or, otherwise, the chance of happening is $1 - (\frac{5}{6})^x = \frac{1}{2}$, $\therefore$ $(\frac{5}{6})^x = \frac{1}{2}$. Hence, x log. $\frac{5}{6} =$ log. $\frac{1}{2}$, and $\therefore$ $x = \dfrac{\log.\ 2}{\log.\ 6 - \log.\ 5} = 3\,.\,8$ nearly, so that the chance is less than $\frac{1}{2}$ in three trials, and greater than $\frac{1}{2}$ in four trials; in four trials a person may safely undertake to do it.

280. The probability of an event happening in one trial being known, required the probability of its happening once, twice, thrice, &c. *exactly*, in n trials.

The probability of its happening in any one trial being expressed by $\dfrac{a}{a+b} = p$, and of its failing by $q = 1 - p$; then the probability of success on any specified trial and failure on the other $(n-1)$, is pq^{n-1}; and since this may happen in n ways, any one of which may be specified, the probability of the event happening in some one of these and failing in the rest is npq^{n-1}. The probability of the event occurring twice in two specified trials, and failing in the other $(n-2)$ trials, is p^2q^{n-2}; now there are (Art. 278), $\dfrac{n(n-1)}{1\,.\,2}$ ways in which the event may happen twice, and of course fail $(n-2)$ times in n trials; $\therefore$ $\dfrac{n(n-1)}{1\,.\,2}p^2q^{n-2}$ expresses the probability that it will happen twice in n trials. Similarly $\dfrac{n(n-1)(n-2)}{1\,.\,2\,.\,3}p^3q^{n-3}$ is the probability that it will happen exactly three times in n trials; and generally

$$\frac{n(n-1)\ldots(n-r+1)}{\underline{|r}}p^rq^{n-r} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is the probability that the event happens exactly r times in n trials, and

$$\frac{n(n-1)\ldots(n-r+1)}{\underline{|r}}p^{n-r}q^r \quad . \quad . \quad . \quad . \quad . \quad (2)$$

that it fails. These several terms want but p^n to form the complete development of a binomial; but p^n expresses (Art. 279) the probability of the *same* event happening each time upon n trials; hence the several terms of the expanded binomial $(p+q)^n$ well express the respective probabilities of the event happening exactly n times, $n-1$ times, $n-2$ times, &c. on n trials; and the greatest term in the series (Art. 275) will give the highest probability of success and failure in n trials. Now $\frac{a}{a+b} = p$, and $\frac{b}{a+b} = q$, and putting $n = m(a+b)$, a, b, and m being whole numbers, the expression, $\frac{(n+1)x}{a+x} = \frac{n+1}{\frac{a}{x}+1}$ Art. 271, becomes in this case $mb + \frac{b}{a+b}$; then the most probable case is that of r failures, and $n-r$ successes, where r is the greatest whole number in $mb + \frac{b}{a+b}$. We have therefore $r = mb$, and $n - r = ma$. Hence, $n-r$ and r, the most probable cases of success and failure, are proportional to a and b (Art. 277), the probabilities in a single trial.

The probability of the *event happening at least* r times in n trials is,

$$p^n + np^{n-1}q + \frac{n(n-1)}{1 . 2}p^{n-2}q^2 \ldots + \frac{n(n-1)(n-2)\ldots(r+1)}{1 . 2 . 3 \ldots (n-r)} p^r q^{n-r}; \quad \ldots\ldots \quad (3)$$

where the coefficient of the general term is the number of combinations of n things taken $n-r$ at a time (Art. 258); for if the event happen every time, or fail only once, twice ... $(n-r)$ times, it will happen r times, so that the sum of the terms above gives the probability that the event will happen at least r times.

Thus, if the chances of throwing three aces in five throws with a single die, or in a single throw with five dice, be required, we have $p=\frac{1}{6}$, $1-p=\frac{5}{6}=q$, $n=5$, and applying (1), we have $\frac{5\,.\,4\,.\,3}{1\,.\,2\,.\,3}\,.\left(\frac{1}{6}\right)^3\,.\left(\frac{5}{6}\right)^2=\frac{250}{7776}$; the chance of throwing at least 3 aces is (3) $p^5+5p^4q+\frac{1}{2}\,.\,5\,.\,4p^3q^2=\frac{276}{7776}$; and of throwing sixes at least twice in 4 throws, $p^4+4p^3q+\frac{1}{2}\,.\,4\,.\,3p^2q^2=\frac{171}{1296}$.

To find the probability of throwing an ace once, and not oftener in four throws, we have

$$p=\frac{1}{6},\ q=\frac{5}{6},\ n=4,\text{ and }\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}\,.\,\frac{1}{6}\,.\left(\frac{5}{6}\right)^3=\frac{125}{324}.$$

281. If the probability of a person's success in regard to any event be p, and s be the advantage or amount that will accrue to him if he be successful, then the value of his hope, or *his expectation*, is sp.

282. The doctrine of probabilities is applied to many practical subjects, such as the expectancy of life and annuities, the estimation of the veracity of witnesses, &c., on which it would be out of place to enter in an elementary treatise. We shall give only one or two examples.

From bills of mortality kept at different places, tables have been constructed, shewing how many persons of any given age die in each year. Thus, Halley's tables shew that of 586 persons aged 22, 579 reach the age of 23; the Carlisle tables that of 6335 aged 14, 6047 reach 21, &c.

To find the probability that a persons of a given age will live one year, or any number of years.

Let a be the number in the tables of the given age, and b the number surviving; then $\frac{b}{a}$ is the probability of life for one year, and (Art. 277) $\frac{a-b}{a}$ that of death. This gives by Halley's tables $\frac{579}{586}$, and $\frac{7}{586}$, or $\frac{83}{84}$, and $\frac{1}{84}$ nearly for the age 22. By the Carlisle tables $\frac{6047}{6335}$ and $\frac{288}{6335}$ are the probabilities that a person of 14 will live to reach 23, or die before that time. In the same way $\frac{b}{a}, \frac{c}{a} \ldots \frac{n}{a}$ are the probabilities of life for 1, 2 . . . . n years, $b, c, \ldots n$ being the number left at the end of 1, 2 . . . n years, and a as before. Also $\frac{a-b}{a}, \frac{a-c}{a} \ldots \frac{a-n}{a}$ are the probabilities of death in 1, 2 . . . n years. The probability that one person A will live a year, being called $\frac{1}{m}$, and that of another B, $\frac{1}{n}$, the probability of both being alive at the end of the year is $\frac{1}{mn}$ (Art. 279); and the probability that they will both die, $\frac{(m-1)(n-1)}{mn}$; the probability that *both* will not die is, $1 - \frac{(m-1)(n-1)}{mn} = \frac{m+n-1}{mn}$. And $\frac{1}{p}$ and $\frac{1}{q}$ being the respective probabilities that each will live y years, $1 - \frac{(p-1)(q-1)}{pq} = \frac{p+q-1}{pq}$ is the probability that one at least will be alive at the end of y years.

On these probabilities of life the calculations of life assurance and annuities are founded. Suppose De Moivres' hypothesis adopted, which is true for middle life, that out of 86 persons born, one dies every year till they are all gone; an assurance office would assume the duration of life for a number of persons aged 20, to be 66 years, and charge for assuring £100 to be paid at death such a yearly premium as, with interest for 66 years, would produce £100, and such additional profit as might be deemed necessary for their own advantage and safety under the circumstances.

If A represent the amount of £1 for one year, that is, £1 with its interest, then $\frac{1}{A}, \frac{1}{A^2}$ &c., is the present value of £1, to be paid in 1, 2, &c., years, and the present value of the annuity of £1 will be

$$\frac{65}{66.A} + \frac{64}{66.A^2} \text{ \&c} ; \ \frac{65}{66}, \frac{64}{66}, \text{ \&c.}$$

being the chances of life for 1, 2, &c. years, in the case of a person whose age is 20. This subject is however viewed differently by writers on Algebra, and the practice of assurance offices varies greatly.

EXERCISES.

1. Required the probability that a person will draw first a white ball and then a black, if six white and five black balls be placed at random in a box.

2. On the same supposition, what is the probability of drawing first a white ball and then two black?

3. In how many trials will the probability of throwing sixes with two dice amount to $\frac{1}{2}$?

4. Required the probability of throwing an ace three times and not oftener in four throws.

5. There are four white balls and three black placed at random in a line; find the probability of the extreme balls being both black.

6. Four white and six black balls are placed at random in a box; find the probability that out of five drawn, two, and two only, shall be white.

7. A box contains three white and seven black balls; required the chance of drawing a white ball in each of three successive trials; the ball not being replaced.

8. On the same supposition, but replacing the balls

after drawing; find the chance of drawing white three times in succession.

9. There are two urns, in one two white balls and one black, in the other, four white and one black; required the probability of drawing a white ball from one of them, by a person from whom both are concealed.

10. A speaks truth three times out of four; B four times out of five; C six times out of seven; required the probability of the truth of what A and B agree in asserting, but which C denies.

XXVIII.—SOLUTION OF GEOMETRICAL PROBLEMS.

283. A line may be represented by a number, expressing how many linear units the line contains. Thus, the line AB

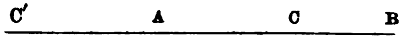
may be represented by any number, as 7; meaning that the line contains 7 inches, 7 feet, 7 miles, or 7 linear units of any kind. In the same manner, the line AB may be represented by any letter, as a; the letter being understood to denote a number of linear units. Also, if the two adjacent sides of a rectangle be represented by any numbers or letters, the product of those numbers or letters will express the number of superficial units in its area: thus, if a and b represent the adjacent sides, ab will represent the area. Hence, if a be the side of a square, its area will be represented by a^2. In the same manner, if a, b, c, represent the number of linear units in the length, breadth, and thickness, respectively, of a solid, the number of its solid units will be expressed by the product abc. Conversely, any single letter may be represented by a line, the product of two letters by a rectangle, and the product of three letters by a solid.

A rectangle *may*, however, be represented by a single letter as a; and if one side of such rectangle be represented by x, then the other will be $\frac{a}{x}$. Also if a represent the area of a square, $\sqrt{a}$ will represent the side. Conversely, it is not *necessary* that every product of two factors should be geometrically represented by a rectangle, and every product of three factors, by a solid: for ab may represent a line, which is a times the line b, that is, which contains a times as many linear units; and abc may express a rectangle, which is a times the rectangle bc. And as geometry is only concerned with the three dimensions of length, breadth, and thickness, it is plain, if a product consist of $3 + n$ factors, they cannot all denote lines; and n of them, at least, must be taken as forming a coefficient to the geometrical magnitude expressed by the others.

The directions that have been given in Art. 158, for the solution of algebraic problems, are applicable also to the solution of geometrical problems. In order to translate the conditions of a problem into equations, the problem must be supposed to be resolved, and relations sought out between the given and required parts; for the purpose of extending these relations, and discovering new ones, additional lines may be drawn, or figures constructed. For this, however, or for the method of translating the conditions into algebraic language, no general rule can be laid down.

PROBLEMS.

1. To divide a straight line, so that the rectangle contained by the whole and one part, shall be equal to the square of the other part. (EUCL. II. 11.)



Let AB $= a$, AC $= x$ $\therefore$ BC $= a - x$; then, by the question, $a(a-x) = x^2$; whence $x^2 + ax = a^2$. Hence we have (Art. 145)

$$x = -\frac{a}{2} \pm \sqrt{\frac{5a^2}{4}};$$

or

$$x = -\frac{a}{2} \pm \frac{a\sqrt{5}}{2} = \frac{a(\pm\sqrt{5}-1)}{2}.$$

Here, if the sign + be taken, x is positive (Art. 154), and less than a; but if the sign − be taken, it is negative, and greater than a: that is to say, the conditions are fulfilled, not only by a point C within the line, as the enunciation contemplated, but also by a point C′ on its continuation. The position of the point was supposed to be such that, in order to find it, something was to be taken from AB; whereas the point may also have such a position that, to find it, something must be added to AB (Arts. 5, 6). To answer this new condition, the question may be easily modified thus:

In a given straight line, or its continuation, to find a point such that the rectangle contained by the given line and the segment between one of its extremities and the required point, may be equal to the square of the segment between its other extremity and the same point.

If the value of x be put under the form $\sqrt{\left(a^2 + \frac{a^2}{4}\right)} - \frac{a}{2}$, it is plain that we shall have (EUCL. II. 11)

$$\sqrt{\left(a^2 + \frac{a^2}{4}\right)} - \frac{a}{2} = \text{EF} - \text{EA} = \text{AH}.$$

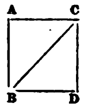
If a square be described on EF + EA = CF, instead of upon EF − EA = AF, and upon the opposite side from AG, BA produced will meet a side of the square in a point H′, which will give $AB \,.\, BH' = AH'^2$: this construction answers to the negative value.

If we had at first supposed the point to have also the position C′, and put BC, or $BC' = x$, we should have had $AC = a - x$, $AC' = x - a$: then $(a-x)^2 = ax$, or $(x-a)^2 = ax$, either of which would give

$$x = \frac{3a}{2} \pm \sqrt{\frac{5a^2}{4}} = \frac{3a}{2} \pm \frac{a\sqrt{5}}{2};$$

both positive values.

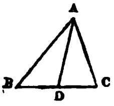
2. Given the difference of the perimeter and diagonal of a square, to find the side and diagonal.



Let one side $AB = x$; then perimeter $= 4x$: also let d = difference of perimeter and diagonal; then $BC = 4x - d$.

Then (Eucl. I. 47) $2x^2 = (4x - d)^2$; hence, $x = \dfrac{d}{4-\sqrt{2}} = \dfrac{d(4+\sqrt{2})}{14}$ (Art. 117). But $BC = 4x - d = x\sqrt{2}$; therefore $BC = \dfrac{d(4+\sqrt{2})\sqrt{2}}{14} = \dfrac{d(2\sqrt{2}+1)}{7}$.

3. Given the line bisecting the vertical angle of a triangle, and the segments into which it divides the base, to find the sides.

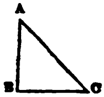


Let $AD = a$, $BD = b$, $DC = c$, $AB = x$, $AC = y$.

Then (Eucl. VI. 3) $b : c :: x : y$, $\therefore cx = by$, and $x = \dfrac{by}{c}$. Again (Eucl. VI. B), $xy = bc + a^2$, $\therefore x = \dfrac{bc + a^2}{y}$. Hence, $\dfrac{by}{c} = \dfrac{bc + a^2}{y}$; whence

$y = \sqrt{\frac{bc^2 + a^2c}{b}}$, and therefore $x = \frac{b}{c}\sqrt{\frac{bc^2 + a^2c}{b}} = \sqrt{\frac{b^2c + a^2b}{c}}$.

4. In a right-angled triangle, given the difference of the legs, and the difference of the hypotenuse and one leg, to find the hypotenuse and legs.



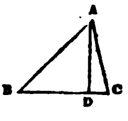
Let CB − BA = a, CA − AB = b, AB = x. Then $x^2 = (x + b)^2 - (x + a)^2$, or $x^2 + 2(a - b)x = b^2 - a^2$; whence (Art. 145),

$$x = \frac{2b - 2a \pm \sqrt{(8b^2 - 8ab)}}{2} =$$

$b - a \pm \sqrt{(2b^2 - 2ab)}$ = AB; and therefore BC = $b \pm \sqrt{(2b^2 - 2ab)}$, and AC = $2b - a \pm \sqrt{(2b^2 - 2ab)}$.

When the triangle is isosceles, $a = 0$, and $x = b(1 \pm \sqrt{2})$, which is obviously greater than the other value of x; and therefore, of all right-angled triangles upon the same hypotenuse, the isosceles is the greatest.

5. Given the perpendicular of a triangle, the difference of the segments of the base, and the ratio of the sides, to determine the triangle.



Let AD = a, BD − DC = d, and AB : AC : : 3 : 2. Also, let AB = x, AC = y, BC = z. Then, $x : y :: 3 : 2$; therefore

$$x = \frac{3y}{2},\ y = \frac{2x}{3}.$$

Now (Eucl. VI. K),

$$dz = \left(x + \frac{2x}{3}\right)\left(x - \frac{2x}{3}\right) = x^2 - \frac{4x^2}{9}\text{; hence, } x^2 = \frac{9dz}{5}.$$

Also (Eucl. I. 47),

$$x^2 = a^2 + \left(\frac{z + d}{2}\right)^2 = \frac{4a^2 + d^2 + 2dz + z^2}{4} = \frac{9dz}{5};$$

therefore, clearing of fractions, and transposing,

$$5z^2 - 26dz = -20a^2 - 5d^2,$$

which gives $$z = \frac{26d \pm \sqrt{(676d^2 - 400a^2 - 100d^2)}}{10};$$

by reduction, $$z = \frac{26d \pm 4\sqrt{(36d^2 - 25a^2)}}{10}.$$

Now, $x^2 = \frac{9dz}{5}$, therefore we have

$$x = \pm\sqrt{\frac{117d^2 \pm 18d\sqrt{(36d^2 - 25a^2)}}{25}} =$$

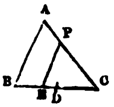
$$\pm \tfrac{3}{5}\sqrt{13d^2 \pm 2d^2\sqrt{(36d^2 - 25a^2)}}:$$

but $y = \frac{2}{3}x$, therefore

$$y = \pm \tfrac{2}{5}\sqrt{13d^2 \pm 2d^2\sqrt{(36d^2 - 25a^2)}}.$$

6. To bisect a triangle by a straight line drawn from a given point in one of its sides.

Let ABC be the given triangle, and P the given point; bisect BC in D, and make (EUCL. VI. 12) PC : CD : : AC : CE; then PC . CE = AC . CD = ½AC . CB, therefore AC . CB = 2PC . CE. But (Thomson's EUCL. VI. 23, *Note*) AC . CB : PC . CE : : ACB : PCE; therefore ABC = 2PCE, or PE bisects the triangle.

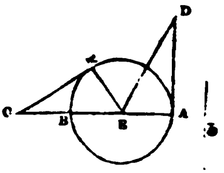


Now, to determine the point E, let BC = a, AC = b, CP = d, and CE = x; then $d : \frac{1}{2}a : : b : x$: hence $x = \frac{ab}{2d}$. When $d = \frac{1}{2}b$, $x = a$; or E coincides with B (EUCL. I. 38).

7. To produce the diameter of a given circle, so that the tangent to the circle from the extremity of the produced part shall have a given ratio to a given line.

Let $m : n$ be the given ratio, and b the given line; from

A draw AD perpendicular to AB; and determine AD by making $m : n :: b : \mathrm{AD} = b\frac{n}{m}$; join ED, and make EC = ED; draw the tangent CF, and join FE: then (Thomson's Eucl. I. 47, *Cor.* 5) $\mathrm{AD} = \mathrm{CF} = \frac{bn}{m}$. Let now the radius $= r$, $\mathrm{BC} = x$; then $\mathrm{CF} = \frac{bn}{m}$



$$= \sqrt{(x+r)^2 - r^2}\,;\ \therefore\ x^2 + 2rx = \frac{b^2n^2}{m^2}:$$

$$\text{whence, } x = \frac{-2r \pm \sqrt{\left(4r^2 + 4\frac{b^2n^2}{m^2}\right)}}{2} = \frac{\pm\frac{2}{m}\sqrt{(r^2m^2+b^2n^2)} - 2r}{2}$$

$$= \frac{\pm\sqrt{(r^2m^2 + b^2n^2)} - rm}{m}.$$

If $m = n$, $\mathrm{CF} = b$, and $x = -r \pm \sqrt{(r^2 + b^2)}$; and if also $b = 2r$, $x = (\pm\sqrt{5} - 1)r$.

EXERCISES.

1. Given the base, b, and the altitude, a, of a triangle, to find the side of the inscribed square.

2. Given the difference, d, between the diagonal of a square, and one of its sides, to determine the square.

3. Given one side, a, of a right-angled triangle, and the sum of the hypotenuse and other side, b, to find the hypotenuse and other side.

4. Given the sum, s, of the perimeter and diagonal of a square to find the side.

5. Given the ratio, $m : n$, of the two sides of a triangle, and the segments, a and b, of the base, made by a perpendicular, to find the two sides.

6. To divide a given straight line, a, so that the rectangle of the two parts may be equal to a given square, b^2.

7. Given the breadth, a, of a street, and the perpendicular heights, b, c, of the houses on opposite sides, to find a point in the street, so that one extremity of a ladder being placed at it, the other will reach to the top of each house.

8. Given one side a, of a right-angled triangle, and the alternate segment, b, of the hypotenuse, to find the whole hypotenuse and the other side.

9. Given the hypotenuse, a, of a right-angled triangle, and the difference, d, between the other two sides, to find the sides.

10. Through a given point in a given circle, to draw a chord of a given length, a.

11. In a rectangle, given a and b, the difference between the diagonal and each of the sides, to find the sides and diagonal.

12. Given two sides, a and b, of a triangle, and the difference, d, of the segments of the base, made by a perpendicular, to find the base.

13. The base and perpendicular of a triangle are known; required the sides, 1st, when their difference is given; and 2d, when their ratio is given.

14. Given the chords of two arcs; required an expression for the chord of their sum.

15. Given the radius r of a circle, to find the sides of the inscribed and circumscribed regular pentagons and decagons.

16. Given, the side of the square, and the radius of the circle, inscribed in a right-angled triangle, to determine the triangle.

ANSWERS.

I.—P. 13.

1. 3. 2. 1, 4, 0, − 1, − 3.
3. 5, 122, 899, 3, 52, 230, 0, 462, 9146, −2, − 6, 1120.
4. 4, − 5. 5. 5, 108, 27, 455, − 1, 0, $-\frac{2}{3}$, $-\frac{25}{24}$.
6. 7, 0, 60, 2112. 7. 252, 2096, −175.
8. 100, 64, 196, $\frac{25}{36}$. 9. 4, 4, 9, $\frac{1}{144}$. 10. 24, 5040.
11. 180, 455. 12. 48, 35, 24, 15, 8.
13. 110, 970, 1692. 14. 0. 15. 9. 16. 1.
17. 13. 18. − 5. 19. − 6. 20. 1.
21. 54. 22. $\frac{1}{4}$. 23. 144. 24. 0, 0, 0, 0, 120, 720.
25. *Any two numbers whose sum is* 10. 26. 1.
27. 5, 26, 44. 28. $8a^2$. 29. $2a^3$. 30. $6x^2 - b^2$.
31. $10 = 10$. 32. $3\frac{1}{2} = 3\frac{1}{2}$. 33. 9. 34. $y = 8$.
35. 0. 36. $\frac{1}{20}$ in both cases. 37. 41.
38. 3, 18, 69. 39. 22, $\frac{7}{16}$, $\frac{3}{4}$, 1, 0. 40. 21, $\frac{81}{256}$.
41. 14, 13. 42. 14, 4, $23\frac{1}{8}$. 43. 492.
44. 63, 14, 120. 45. 96, 4, 2226, 4, 32.
46. 84, 600. 47. 10. 48. 64. 49. 3, 5, 5.
50. $\frac{4}{15}$. 51. 36. 52. $23\frac{1}{3}$. 53. $\frac{21}{15} = \frac{7}{5}$.
. . . . . . 64. 1° 17′ less than nothing, or − 1° 17′.

II.—P. 20.

1. $12ax - 11b^2y$. 2. $2a - 3x$. 3. $a + 5x^2 + 4b^2$.
4. $a + b$. 5. $3ax - 2xy + 7x^2y^2$. 6. $-3a + 2b + 4x - 2y$.

7. $-4a + b + 8c - 11d - 3ey$. 8. $4x(a + b) - 5\sqrt{x} - 5$.
9. $(a + b - d)x + (b + d - 3)y + 4x^2$.
10. $-9ax - 5b - 5$. 11. $a - 4$. 12. $9 - 3x + x^2$.
13. $6 + 4\sqrt{7}$. 14. $5a + 2x + 3b + 2d - c$.
15. $2(a + b)\sqrt{x} + 2(a + b)x - 4d\sqrt{(x^2 - y^2)} + (3mn + 1)4p$.
16. $6\sqrt{x} - \sqrt{(a + b)}$.
17. $(4a + m)\sqrt{x} - \sqrt[3]{y} + (2a - b)x + 3ax^2 + 4$.

III.—P. 24.

1. $20a - 9x + 10y - 23$. 2. $2a + 10b + 2c + 13\sqrt[4]{x}$.
3. $7a\sqrt{(x^2 + y)} + 3a^2x + 1$. 4. $b + b^2 - 2c - 11$.
5. $(a - e)x^2 + (b + f)x + (c - g)xy - d + h$.
6. $-5ax^2 - 8\sqrt{b} + 4$. 7. $ab(x - 1) + 2x^2 + 4\sqrt[3]{y} - 2\sqrt{y}$.
8. $5(x + 1)$. ... 55. $2b$. 56. $-2x - 2$.
57. $a + 2b + c$. 58. x. 59. $y^2 + 3xy - x^2$.
60. $a\sqrt{(x - y)}$. 61. $a - 3\sqrt{x}$. 62. b.
63. $2a^2 - a + b - 2b^2 + x$. 64. $7a - 5b$. 65. $-2\sqrt{x}$.
66. x. 67. $6a$. 68. $2a - b - d$.

IV.—P. 35.

63. $24a^3 + 28a^2x - 36ax^2 + 8x^3$.
64. $a^{m+1} + 2a^mx + 2a^{m-1}x^2 + 2a^{m-2}x^3 + a^{m-3}x^4$.
65. $2a^{2m} - 7a^mb^n + 6b^{2n}$. 66. $x^4 + 64$.
67. $1 - 6x^5 + 5x^6$. 68. $a^5 + 32b^5$.
69. $35a^2b^2 - 69a^2bc - 18ab^2c - 54a^2c^2 + 78abc^2 - 8b^2c^2$.
70. $x^7 - 6x^6 + 16x^5 - 4x^4 - 26x^3 + 5x^2 - 4$.
71. $4x^6 - 5x^5 + 8x^4 - 10x^3 - 8x^2 - 5x - 4$.
72. $x^6 - 22x^4 + 60x^3 - 55x^2 + 12x + 4$.
73. $a^7 - 7a^6 + 21a^5 - 17a^4 + 25a^3 + 8a^2 - 2a - 4$.
74. $2a^4x^4 + a^5x^6 - 8a^6x^8 + 12a^7x^{10} - 22a^8x^{12} + 15a^9x^{14}$.
75. $21x^3y^8 - 3x^4y^7 + 20x^5y^6 - 37x^6y^5 + 9x^7y^4 - 10x^8y^3$.
76. $21a^5x^3 - 2a^4x^4 - 4a^3x^5 + 5a^2x^6 - 2ax^7$ 77. $6x^4 - 96$.

78. $8x^4 - 4x^3 - 70x^2 + 129x - 63$.

79. $a^{11}x^6 - 2a^9x^5 - 6a^7x^4 - 2a^5x^3 + a^3x^2$.

80. $4v^5 - 7v^6 - 10v^7 + 19v^8 - 6v^9$.

81. $v^6 - 3v^5 + 3v^4 - 9v^3 + 12v^2 + 12v - 16$.

82. $15a^7 + a^5b^2 - 31a^4b^3 - 6a^3b^4 + 11a^2b^5 + 10ab^6$.

. . . 87. $x^4 - 15x^2 - 10x + 24$.

88. $x^4 + 2x^3 - 85x^2 - 86x + 1680$.

89. $-a^4 - b^4 - c^4 + 2(a^2b^2 + a^2c^2 + b^2c^2)$. 90. $16a^4 - b^4$.

V.—P. 46.

. . . 66. $x^2 - 6x + 9$. 67. $a - b$.

68. $a^n + a^{n-1}x + a^{n-2}x^2 + \&c$. 69. $1 - 3x + 3x^2 - x^3$.

70. $2x^2 + 5x - 7$. 71. $1 + n$. 72. $x^3 + ax^2 + a^2x + a^3$.

73. $x^3 - ax^2 + a^2x - a^3$.

74. $x^{q(p-1)} + x^{q(p-2)} + x^{q(p-3)} + \ldots + x^{2q} + x^q + 1$.

75. $x^3 + ax^2 + a^2x + a^3 + \dfrac{2a}{x-a}$.

76. $2x^3 + 4x^2 + 8x + 16$.

77. $a^5 + a^4x + a^3x^2 + a^2x^3 + ax^4 + x^5$.

78. $n^4 + n^3p + n^2p^2 + np^3 + p^4$.

79. $n^4 - n^3p + n^2p^2 - np^3 + p^4$.

80. $x^8 - ax^7 + a^2x^6 - a^3x^5 + a^4x^4 - a^5x^3 + a^6x^2 - a^7x + a^8$.

81. $x^5 + x^4 + x^3 + x^2 + x + 1$.

82. $a^{n-1} - a^{n-2}b + a^{n-3}b^2 - a^{n-4}b^3 + \&c$.

83. $2a + 5b - 7c + 3d$. 84. $2a^3x^2 + 3a^4x - a^5$.

85. $3x^2 + 4xy - 2y^2$. 86. $-2a^2 + 8ax - 5x^2$.

87. $-8y^3 + 4xy^2 - 3x^2y + 7x^3$. 88. $x - 3x^3 + 5x^5 - 7x^7$.

89. $4a^3b - 3ab^3 - b^4 + \dfrac{ab^5 - 2b^6}{3a^2 + 2ab - b^2}$.

90. $a^5 - 2a^4x + 3a^3x^2 - 3a^2x^3 + 2ax^4 - x^5$.

VI.—G. C. M, P. 63.

1. $x - a$. 2. $a^2 - x^2$. 3. $2a + 3x$. 4. $3a^2 + 2ab - b^2$.

5. $2x^3 - 4x^2 + x - 1$. 6. $y + 4$. 7. $7x^2 - 3ax + a^2$.

8. $5a^2 - 3a + 4$. 9. $x^2 - 2x + 3$. 10. $3x^3 + 4x^2 + 3x - 2$.
11. $x^3 + x^2 - 5x + 3$. 12. $x^2 - 1$. 13. $2x^2 - 2ax$.
14. $x - 5$. 15. $4x^2 - 9x + 6$. 16. $x^2 + 2x + 3$.
17. $q + 6$. 18. $a - x$.
19. $a - \frac{1}{2}\sqrt{(a + 1)}$. 20. $6a - 21$.

L. C. M., P. 65.

1. $120x^4$. 2. $x^2 - 1$. 3. $1890abx$.
4. $8x^4 - 18x^3 + 15x^2 - 7x - 3$. 5. $x^4 + ax^3 - a^3x - a^4$.
6. $x^5 - 3x^4 - 49x + 147$. 7. $a^5 - a^4x - ax^4 + x^5$.
8. $x^6 - a^6$. 9. $6x^3 - 25x^2 + 23x - 6$.
10. $x^6 - 2x^5 + x^4 - x^2 + 2x - 1$.

VII.—P. 79.

. . . . 33. $\frac{a^2 - 2ax + x^2}{a^3 + a^2x + ax^2 + x^3}$. 34. $\frac{2x + 3}{3x^2 - 7}$.

35. $\frac{3a^2 + 2x^2}{2a^3 - x^3}$. 36. $\frac{x^{2n} - ax^{2n-1} \ldots - a^{2n-1}x + a^{2n}}{x^{2n} - ax^{2n-1} \ldots a^{2n-1}x + a^{2n}}$.

37. $\frac{x^2 - 7x + 10}{x - 10}$. 38. $\frac{3a^2 + b^2}{4a - b}$. 39. $\frac{x^3 - 2x^2 - x + 2}{4x^2 - 5x - 3}$.

40. $\frac{x^2 + 2x - 15}{3x + 7}$. 41. $\frac{7a^2 - 3bx}{5a + a^4b^2x - 4}$. 42. $\frac{2b - 3ac}{4a^2b^2 - 1}$.

43. $\frac{x^2 - 5x + 6}{3x^2 - 9x + 8}$ . . . 53. $a(b^4 - b^3x + b^2x^2 - bx^3 + x^4) - \frac{c}{b + x}$

54. $x^2 + xy + y^2 + x + y - \frac{y^2}{x - y}$. 55. $1 + b - a$.

56. $2a - 3b^2 + x - 5b$. 57. $3a^3 - 5a^2b + 2ab^2$.

58. $\frac{2x + 1}{b^2x}$, $\frac{b^2(3a - 1)}{b^2x}$. 59. $\frac{a}{6xy}$, $\frac{2by}{6xy}$, $\frac{3cx}{6xy}$.

60. $\frac{3dm}{3m}$, $\frac{m(x + 1)}{3m}$, $\frac{3n}{3m}$. 61. $\frac{(a - b)x}{a^2 - b^2}$, $\frac{(a + b)y}{a^2 - b^2}$, $\frac{z}{a^2 - b^2}$

62. $\frac{6a - 3}{5ax}$, $\frac{3xy}{5ax}$. 63. $\frac{p - q}{a^2 - n^2}$, $\frac{x(a - n)}{a^2 - n^2}$.

64. $\frac{a+x}{b}$. 65. $\frac{7a}{12}$. 66. $\frac{7a}{4b}$, or $\frac{a}{4b}$. . . .

74. $\frac{4pq}{p^2-q^2}$. 75. $\frac{2n-1}{n(n-1)}$. 76. $\frac{1}{abc}(bc+ac+ab)$.

77. $\frac{100a+10b+c}{10^3}$. 78. $\frac{2b}{a^2-b^2}$. 79. $-\frac{1}{1+x}$.

80. $\frac{x}{b}$. 81. $\frac{2x-11}{x^2-11x+30}$, or $\frac{1}{x^2-11x+30}$.

82. $\frac{1}{1-x^2}$, in both cases. 83. $\frac{-3}{x^2-7x+10}$.

84. Denom. $(x+a_1)(x+a_2)(x+a_3)$; num. $(n_1+n_2+n_3)x^2+\{n_1(a_2+a_3)+n_2(a_1+a_3)+n_3(a_1+a_2)\}x+n_1a_2a_3+n_2a_1a_3+n_3a_1a_2$.

85. $\frac{x^2}{(x+1)(x+2)(x+3)}$. 86. $\frac{1}{x+2}$.

87. $\frac{169a+77}{105}$. 88. $a-2x$. 89. 0. 90. $\frac{1-a^2}{a+7}$.

91. $\frac{19xy-9y^2}{3y}$. 92. $\frac{19}{12}x$. 93. $\frac{2ab+5a}{2b}$.

94. $\frac{5}{x+y}$. 95. $\frac{a(4b-3)}{4b}$. 96. $\frac{2}{b-y}$.

97. 0. 98. 0. 99. $\frac{6}{x-m}$. . . . 105. $\frac{3q-1}{q-2}$.

106. $\frac{a^2+b^2}{(a+b)^2}$. 107. $\frac{x-1}{b+4}$. 108. $a^2-\frac{1}{a^2}$.

109. $\frac{5a^2}{4x^2}+\frac{ab}{6x}-2b^2$. 110. $\frac{2}{9}-4a+18a^2$.

111. $\frac{4ax}{a^2-x^2}$. 112. $\frac{a^{n+1}-ax^n-a^nx+x^{n+1}}{2(b^{2n}-x^{2n})}$.

113. $\frac{3x}{4y}$. 114. $\frac{x^2-11x+28}{x^2}$.

115. $qrs+prs+pqs-s$. 116. $7x$. 117. $\frac{mq}{np}$.

118. $\frac{m-q}{m+q}$. 119. $\frac{4ab}{5xy}$. 120. $\frac{5x}{a}$.

121. $\frac{3}{2}$. 122. $\frac{a^2+1}{a}$. 123. $\frac{12b+6}{8b+3}$.

124. $\frac{a^2+b^2}{(a-b)^2}$. . . . 131. $\frac{3a^2+4a-2}{a}$.

132. $\frac{x^2+2x+1}{x^2+5x+6}$. 133. $x^3+\frac{1}{x^3}$. 134. $\frac{a^2+b^2}{2ab}$.

135. $\frac{x}{1+x}$. 136. $\frac{ad}{bd+c}$. 137. $\frac{1}{x}$.

138. $x^2-7x+12$. 139. $\frac{a^2+a}{a^2+1}$.

140. $\frac{6x^2-4ax-a^2}{6(x-a)}$. 141. $\frac{1+x^2}{1+x+x^2}$.

142. $\frac{1+x}{1+2x}$. 143. $\frac{4}{3x}$. 144. $\frac{x+3x^2+2x^3+x^4}{1+4x+3x^2+2x^3}$.

145. b. 146. 1. 147. 1. 148. $\frac{a^2+x^2}{2ax}$.

149. $\frac{4a^2x}{x^4-a^4}$. 150. $\frac{bc+ca+ab}{bc+ca-ab}$.

P. 89.

1. $4(1+a+a^2+a^3$ &c.$)$ 2. $1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}$, &c.

3. $a(1-x+x^2-x^3+$ &c.$)$ 4. $a\left(\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\frac{1}{x^4}+\text{\&c.}\right)$

5. $\frac{a}{b}+\frac{(a-b)x}{b^2}+\frac{(a-b)x^2}{b^3}+$ &c.

6. $1-\frac{1}{x}+\frac{3}{x^2}-\frac{3^2}{x^3}+\frac{3^3}{x^4}-$ &c. 7. $\frac{1}{x^2}+\frac{2}{x^3}+\frac{3}{x^4}+\frac{4}{x^5}+$ &c.

8. $1+x+2x^2+2\,.\,4x^3+2\,.\,4^2x^4$, &c.

9. $1+2x+3x^2+4x^3+5x^4+$ &c.

10. $\frac{1}{x}-h\,.\,\frac{1}{x^2}+h^2\,.\,\frac{1}{x^3}-h^3\,.\,\frac{1}{x^4}+$ &c.

VIII.—P. 97.

. . . . 16. $\frac{x(1-x)}{(1-x^2)^{\frac{1}{2}}}$.

17. $a^6-36a^5+540a^4-4320a^3+19440a^2-46656a+46656$.

18. $625+2000x+2400x^2+1280x^3+256x^4$.

19. $1+nx+\frac{n(n-1)}{1\,.\,2}x^2+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^3\ldots+x^n$.

20. $\frac{x}{a}-\frac{bx}{a^2}+\frac{b^2x}{a^3}-\frac{b^3x}{a^4}+$ &c. *See* Art. 81.

21. $\frac{1}{a^3}\left(1-\frac{6b}{a}+\frac{24b^2}{a^2}-\frac{80b^3}{a^3}\text{ \&c.}\right)$

22. $x^2+\frac{1}{x^2}+2$. 23. $\frac{1}{\sqrt[3]{x^2}}$

24. $x^{10}+15x^8y^2+90x^6y^4+270x^4y^6+405x^2y^8+243y^{10}$.

25. $\frac{1}{100}$; $\frac{1}{32}$; 49; 375. 26. $\frac{2}{a^3}\left(1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}+\text{\&c.}\right)$

27. $1+(n+1)x+(n+1)\frac{n}{2}x^2+(n+1)\frac{n(n-1)}{1\,.\,2\,.\,3}x^3+$
$\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3\,.\,4}x^4+\ldots+x^{n+1}$.

28. $495a^{16}x^8$. 29, $12870a^8x^8$. 30. 56.

31. $1+nx+\frac{n(n+1)}{1\,.\,2}x^2+\frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}x^3+$ &c.

32. $a^{10}-10a^9+45a^8-120a^7+210a^6-252a^5+210a^4-120a^3$
$+45a^2-10a+1$. 33. $1+3x+6x^2+10x^3+15x^4$ &c.

34. $1-3x^2+6x^4-10x^6$ &c. . . . 49. $\frac{3a}{2(1-x)^{\frac{5}{2}}}$.

53. $729x^6+2916x^5y+4860x^4y^2+4320x^3y^3+2160x^2y^4$
$+576xy^5+64y^6$.

55. $x^{-2}-2x^{-1}y^{-1}+y^{-2}$; $\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}$.

. . . 58. $(3a^2+2bc)^2+2(3a^2+2bc)\times 4x^3+(4x^3)^2=9a^2+$
$12a^2bc+4b^2c^2+24a^2x^3+16bcx^3+16x^6$.

59. $(1+x)^2+2(1+x)(x^2+x^3)+(x^2+x^3)^2$, or Art. 92, p. 96,

$1 + x^2 + x^4 + x^6 + 2(x + x^2 + x^3) + 2x(x^2 + x^3) + 2x^2 \times x^3$; by either method, $1 + 2x + 3x^2 + 4x^3 + 3x^4 + 2x^5 + x^6$.

60. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$; $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$.

IX.—P. 101.

... 6. $\sqrt[12]{x}$. ... 8. $\frac{2a^{\frac{3}{2}}}{3b^2}$. 9. $x^{\frac{1}{12}}$. 10. $\sqrt[4]{y}$. ... 19. $\frac{1}{a^{\frac{1}{2}}}$.

23. x^{n+2}. ... 27. $\sqrt[12]{(a^2 + x^2)}$. ... 31. $x^{\frac{1}{24}}$. 33. $\sqrt[120]{a}$.

P. 116.

1. $5a + 6$. 2. $3x + 4y$. 3. $a - \frac{1}{2}x$.

4. $2x^2 - 4x + 2$. 5. $2a^2 + 3ax + x^2$. 6. $\sqrt{x} + \sqrt{\frac{1}{x}}$.

7. $a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5}$ &c. 8. $5x^2 - 3ax + 4a^2$.

9. $x^2 - x + \frac{1}{4}$. 10. $5a^{m-2}x^{n+1}c + a^m x^n - \frac{3a}{x}$.

11. $x^3 - 3ax^2 + 3a^2x - a^3$. 12. $p + qx + rx^2 + sx^3$.

13. $q^2 = \frac{p^2}{4a^2}$, or $p^2 = 4a^2q^2$. 14. $2x - 5$.

15. $\frac{acx^2}{b} - \frac{bx}{c}$. 16. $(a + 1)^{2m}x + 2ca^p$. 17. $x^2 - 2x + 1$.

18. $3x^2 - 2x + 1$. 19. $2x^2 + 4\,ax - 3a^2$.

20. $1 + 2x + 3x^2 + 4x^3$. 21. $a^m - 2ax^n$. 22. $3x^2 + 2x + 7$.

23. $x^2 - x + 2$. 24. $c = \frac{b^2}{3a^3}$, $d = \frac{b^3}{27a^6}$.

25. $mp^2 = qn^2$; $n^2 = 3mp$.

26. $b + \frac{x}{2b} - \frac{x^2}{2\,.\,4b^3} + \frac{3x^3}{2\,.\,4\,.\,6b^5} - \frac{3\,.\,5x^4}{2\,.\,4\,.\,6\,.\,8b^7} +$ &c.

27. $\frac{1}{b} - \frac{x}{2b^3} + \frac{1\,.\,3x^2}{2\,.\,4b^5} - \frac{1\,.\,3\,.\,5x^3}{2\,.\,4\,.\,6b^7} + \frac{1\,.\,3\,.\,5\,.\,7x^4}{2\,.\,4\,.\,6\,.\,8b^9} -$ &c.

28. $1 + \frac{1}{2} - \frac{1}{2\,.\,4} + \frac{1\,.\,3}{2\,.\,4\,.\,6} - \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6\,.\,8} + \frac{1\,.\,3\,.\,5\,.\,7}{2\,.\,4\,.\,6\,.\,8\,.\,10}$ – &c. 29. $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128}$ &c.

30. $1+\frac{1}{n}x-\frac{n-1}{2n^2}x^2+\frac{(n-1)(2n-1)}{2\,.\,3n^3}x^3$ &c.

31. $a^{\frac{2}{3}}\left(1+\frac{2x}{3a}-\frac{x^2}{9a^2}+\frac{4x^3}{9^2a^3}-\frac{4\,.\,7x^4}{9^2.\,12a^4}+\frac{4\,.\,7\,.\,10x^5}{9\,.\,12\,.\,15a^5}\text{ &c.}\right)$

32. $2-\frac{1}{3.2^2}+\frac{1}{3\,.\,6\,.\,2^4}-\frac{5}{3\,.\,6\,.\,9\,.\,2^7}-\frac{5\,.\,8}{3.6.9.12.2^{10}}$ &c.

33. $\frac{a}{c}\left(1-\frac{x^2}{2\,.\,c^2}+\frac{3x^4}{2^3.c^4}-\frac{5x^6}{2^4.c^6}\text{ &c.}\right)$

34. $\frac{1}{a}+\frac{1}{2}\cdot\frac{x^2}{a^3}+\frac{3}{8}\cdot\frac{x^4}{a^5}+\frac{5}{16}\cdot\frac{x^6}{a^7}$ &c.

35. $1-\frac{1}{n}x+\frac{n+1}{2n^2}x^2-\frac{(n+1)(2n+1)}{2\,.\,3\,.\,n^3}$ &c.

36. $1+\frac{1}{5}-\frac{2}{25}+\frac{3}{250}-\frac{21}{2500}$ &c. $=1.1236$ nearly.

37. $a^{\frac{1}{2}}\left(1-\frac{x}{2a}-\frac{x^2}{2\,.\,4a^2}-\frac{1\,.\,3x^3}{2\,.\,4\,.\,6a^3}\right.$ &c.

38. $\frac{1}{a^{-\frac{4}{3}}}+\frac{1\,.\,4x}{3\,.\,a^{-\frac{1}{3}}}+\frac{1.4.x^2}{3.6a^{\frac{2}{3}}}-\frac{1\,.\,2\,.\,4x^3}{3\,.6\,.\,9a^{\frac{5}{3}}}+\frac{1.2.4.5x^4}{3.6.9.12a^{\frac{8}{3}}}-$ &c.

X.—P. 128.

1. $\sqrt{9x^4}$. 2. $\sqrt[3]{(8a^6b^3)}$. 3. $\sqrt[3]{(-27a^3)}$.

4. $\sqrt[n]{(a^{2n}b^{\frac{n}{2}})}$ 5. $\sqrt[5]{(-32n^{10})}$. 6. $\sqrt[3]{27x}$. 7. $(a^{\frac{3}{2}})^{\frac{1}{3}}$.

8. $\sqrt{(\frac{1}{16}.8x)}$. 9. $(4\,.3^{\frac{2}{3}})^{\frac{1}{2}}$. 10. $[(a-x)^{\frac{4}{5}}]^{\frac{5}{6}}=\{\sqrt[5]{(a-x)^4}\}^{\frac{5}{6}}$.

11. $(a^{\frac{6}{7}})^{\frac{7}{10}}$. 12. $(a^{\frac{q}{p}}x)^{\frac{p}{q}}$ 13. $\{9(a+b)\}^{\frac{3}{2}}$; $9^{\frac{3}{2}}=\sqrt{9^3}=27$.

14. $\sqrt[3]{\{a^{\frac{3}{2}}(a+x)\}}$. 15. $3a^2\sqrt{x}$. 16. $8a(a^2+2)^{\frac{1}{3}}$.

17. $5\sqrt{3}$. 18. $9\sqrt{5}$. 19. $7\sqrt{3}$. 20. $3\sqrt[3]{6}$.

21. $\frac{1}{2}\sqrt{3}$. 22. $\frac{2}{3}\sqrt{2}$. 23. $\frac{1}{5}\sqrt{5}$. 24. $\frac{2}{3}\sqrt{3}$.

25. $\frac{1}{b}\sqrt{ab}$. 26. $\frac{a-2x}{np}(ax)^{\frac{1}{2}}$ 27. $6\sqrt{(2x+3y)}$.

28. $\frac{ab}{cd}(ac-bd)^{\frac{1}{2}}$. 29. $\frac{2x^{n+1}}{a}\sqrt[3]{4}$. 30. $(a+b)\sqrt{(3c)}$.

31. $(y-1)\sqrt{(2n)}$. 32. $a\sqrt{b}$. 33. $2\sqrt[3]{a}$.

34. $a\sqrt[3]{(1+b^3)}$. 35. $x\sqrt[3]{(2y^2+bx^2y)}$. 36. $\frac{1}{3}\sqrt{3}$.

37. $\frac{1}{7}\sqrt{14}$. 38. $\frac{1}{3}\sqrt[3]{12}$. 39. $\frac{a-x}{a+x}\sqrt{x}$.

40. $\frac{2}{27}\sqrt[3]{18}$. 41. $(a^3)^{\frac{1}{6}}, (a^4)^{\frac{1}{6}}$. 42. $32^{\frac{1}{20}}, 81^{\frac{1}{20}}$.
43. $\sqrt[12]{a^8}, \sqrt[12]{a^9}$. 44. $\sqrt[3]{a^6}, \sqrt[3]{b^2}$. 45. $(5^{10})^{\frac{1}{15}}, (6^{12})^{\frac{1}{15}}$.
46. $2\sqrt[6]{9}, 3\sqrt[6]{8}$. 47. $\sqrt[6]{(a+x)^4}, \sqrt[6]{(b-x)^2}$.
48. $\sqrt[n]{x^{n^2}}, \sqrt[n]{y}$. 49. $\sqrt[10]{\frac{1}{32}}, \sqrt[10]{\frac{9}{16}}$.
50. $a\sqrt[nm]{a^m}, b\sqrt[nm]{y^n}$. 51. $5\sqrt{5}$. 52. $2\sqrt[3]{5}+5\sqrt{6}$.
53. $2\frac{5}{36}\sqrt{3}$. 54. $9\sqrt{5}$. 55. $5\sqrt[3]{6}, \sqrt[3]{6}$.
56. $8\sqrt{3}+27\sqrt[3]{4}$. 57. $11\sqrt[4]{2}$. 58. $4a\sqrt{(3b)}$.
59. $(3a^2b+5ab)\sqrt{(2ab)}$. 60. $\frac{5}{16}\sqrt[3]{2}, \frac{3}{16}\sqrt[3]{2}$.
61. $9\sqrt[3]{2a}$. 62. $6\sqrt[3]{3}, 2\sqrt[3]{3}$. 63. $a^2\sqrt{x}$.
64. $\frac{1}{4}\sqrt[3]{3}$. 65. $a^2b^{\frac{4}{3}}$. 66. $\frac{1}{2}\sqrt[5]{2}$. 67. $2\sqrt[5]{16}-3\sqrt[3]{2}$.
68. $2\sqrt[4]{2}-4\sqrt[3]{5}$. 69. $8\sqrt[3]{4}, 2\sqrt[3]{4}$. 70. $(a+7c\sqrt{(5c)}$.
71. $\frac{11}{5}\sqrt{15}$. 72. $(28+4a^2x)\sqrt[3]{2}$. 73. $5a(2b)^{\frac{1}{3}}+3ab^{\frac{1}{2}}$.

74. $8\sqrt[3]{2}$. 75. $8+7\sqrt[3]{3}$. 76. 0. 77. $\frac{a}{2c}\sqrt{ab}$.

78. $15\sqrt{(bx)}$. 79. $20^{\frac{1}{10}}$. 80. 48. 81. $a^{\frac{5}{6}}$.
82. $12(a^3b^4)^{\frac{1}{12}}$. 83. $\frac{1}{3}\sqrt[3]{15}$. 84. $\sqrt[12]{648000}$.
85. $30\sqrt{(2+3\sqrt{2})}$. 86. $a+x+(ax+x^2)^{\frac{1}{2}}$.
87. $(a^2-x^2)^{\frac{3}{2}}$. 88. $\frac{1}{10}\sqrt[3]{2}$. 89. 140. 90. $7\sqrt[3]{15}$.
91. $a^{\frac{1}{2}}x^{\frac{1}{3}}$. 92. $8\sqrt{6}$. 93. abc. 94. 4.
95. $5\sqrt[3]{45}$. 96. $(1-x^2)^{\frac{1}{2}}$. 97. $\sqrt[nq]{(x^{mq+np})}$.
98. $(ac+ad^{\frac{1}{2}}+cb^{\frac{1}{2}}+b^{\frac{1}{2}}d^{\frac{1}{2}})^{\frac{1}{2}}$. 99. $44-8\sqrt[3]{4}-30\sqrt[3]{2}$.
100. $x^2-y+2xz^{\frac{1}{3}}+z^{\frac{2}{3}}$. 101. $a^{\frac{4}{3}}+a^{\frac{1}{3}}x-ax^{\frac{1}{3}}-x^{\frac{4}{3}}$.
102. $-a^2-b^2-c^2+2(ab+ac+bc)$. 103. a^2b^6c.
104. x^2+px-q. 105. $1-\sqrt{5}$. 106. $3-17\sqrt{6}$.
107. $42+7\sqrt{30}+2(\sqrt{15}+\sqrt{18})$. 108. a^2-x^2.
109. $42\sqrt{10}-174$. 110. $\sqrt[n]{(a^2-b)}$. 111. $2\sqrt{5}$.
112. $6\sqrt{3}$. 113. 3. 114. $\frac{2}{3}\sqrt[3]{18}$. 115. $\sqrt{2}$.
116. $x^{\frac{1}{3}}$. 117. $2\sqrt{3}$. 118. $(1-x)^{\frac{1}{2}}$.

119. $\sqrt{\frac{n+m}{n-m}}$. 120. b. 121. $5-\sqrt{2}$.

122. $\sqrt[20]{a}$. 123. $\frac{4}{5}\sqrt[3]{3}$. 124. $\frac{61}{72}\sqrt{3}$. 125. $a^{\frac{1}{6}}$.
126. $\sqrt[5]{2}$. 127. $2\sqrt[6]{\frac{16}{3}}$. 128. 6.

129. $a^2 + a^{\frac{3}{2}}x^{\frac{1}{2}} + ax + a^{\frac{1}{2}}x^{\frac{3}{2}} + x^2$. 130. $\sqrt[4]{\frac{b}{a}}$.

131. $\frac{5}{7}\sqrt[m]{x^{m-n}}$. 132. $\sqrt[20]{x}$. 133. 15. 134. $\frac{1}{x^{\frac{1}{4}}}$.

135. $x^{\frac{m-n}{n}}$. 136. $\sqrt{2}$. 137. $a^{\frac{1}{2}}-2b^{\frac{1}{4}}$. 138. $a+\sqrt{b}$.
139. $4\sqrt{6}-\sqrt{3}-7\sqrt{2}$. 140. $4a^{\frac{2}{3}}-a^{\frac{1}{3}}$. 141. $2+\sqrt{3}$.
142. $\sqrt{5}-1$. 143. $\sqrt{5}-\sqrt{2}$. 144. $\frac{1}{2}(\sqrt{6}+\sqrt{2})$.
145. $1+\frac{1}{2}\sqrt{2}$. 146. $4+\sqrt{7}$. 147. $\sqrt[4]{18}-\sqrt[4]{2}$.
148. $3+\sqrt{2}$. 149. $7-3\sqrt{5}$. 150. $\sqrt[4]{20}+\sqrt[4]{5}$.
151. $\sqrt[4]{12}+\sqrt[4]{3}$. 152. $3\sqrt[4]{27}+\sqrt[4]{3}$.
153. $1+\sqrt{2}+\sqrt{5}$. 154. $1+\sqrt{3}$. 155. $1+\sqrt{5}$.
156. a^2-b. 157. $24\frac{1}{4}$. 158. $a-b$. 159. a^2b-c^2d.
160. 41. 161. $2-\sqrt{3}$. 162. $2+\frac{3}{2}\sqrt{2}$.

163. $8+2\sqrt{15}$. 164. $\frac{\sqrt{a}(b-\sqrt{c})}{b^2-c}$. 165. $\frac{1}{10}$.

166. $\sqrt[n]{\frac{2a+2\sqrt{(ab)}+b}{a-b}}$. 167. $\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4}$.

168. $\sqrt{(1-x)}$. 169. $\frac{x+\sqrt{(x^2-a^2)}}{a}$.

XI.—P. 135.

1. $2a$, $2b\sqrt{(-1)}$. 2. $a+c+(b+d)\sqrt{-1}$.
3. $a-2b\sqrt{-1}$. 4. $-ab$. 5. $-\sqrt{xy}$. 6. ab.
7. $-a^3\sqrt{-1}$, a^4. 8. $1+\sqrt{(-1)}$, $1-\sqrt{(-1)}$. 9. $-6\sqrt{6}$.
10. $a^3-3ab^2+(3a^2b-b^3)\sqrt{-1}$. 11. $ac-bd\pm(ad+bc)\sqrt{-1}$.
12. $5a^2+4b^3\sqrt{-1}$, $5a^2-4b^3\sqrt{-1}$; or $4b^3+5a^2\sqrt{-1}$,
and $4b^3-5a^2\sqrt{-1}$.

13. $\frac{2a}{a^2+b^2}$, $\frac{2b\sqrt{-1}}{a^2+b^2}$; $\frac{2(a^2-b^2)}{a^2+b^2}$, $\frac{4ab\sqrt{-1}}{a^2+b^2}$.

14. $\sqrt{a}+\sqrt{b}\sqrt{-1}$, $\sqrt{a}-\sqrt{b}\sqrt{-1}$. 15. 1. 16. $\frac{3}{2}\sqrt{3}$.
17. $\sqrt{-1}$. 18. $6\pm8\sqrt{-1}$, or $8\pm6\sqrt{-1}$.

19. x^4-a^4. 21. 625. 22. $\frac{\sqrt{2}}{2}\sqrt{-1}$.

23. $e^{(a+b)\sqrt{-1}}$. 24. $e^{na\sqrt{-1}}$. 25. 625.

XII.—P. 143.

1. 3. 2. 7. 3. 1. 4. 10, or − 6. 5. $\frac{3}{4}$.
6. 6. 7. 3. 8. 5. 9. 2. 10. 12, or 2.
11. 6, or − 14. 12. 3. 13. 3. 14. 6. 15. $11\frac{1}{2}$.
16. 6. 17. 3. 18. 12. 19. 12. 20. 60.
21. 60. 22. $-\frac{6}{83}$. 23. 60. 24. 24. 25. 10.
26. 7. 27. 7. 28. 8. 29. 4. 30. 11. 31. − 6.
32. $3\frac{6}{13}$. 33. 12. 34. 3. 35. 20. 36. 8.
37. $2\frac{21}{29}$ 38. 10. 39. 3. 40. 5. 41. − 5.
42. − 3. 43. 2. 44. 2. 45. 7. 46. $3\frac{4}{7}$.
47. $4\frac{7}{9}$. 48. $\frac{65}{66}$. 49. 19. 50. $3\frac{3}{8}$. 51. $4\frac{1}{3}$.
52. 2. 53. 4. 54. 1. 55. 7. 56. 1.
57. $4\frac{1}{2}$. 58. 2. 59. $\frac{1}{6}$. 60. 20.

Literal Equations.

61. $\frac{a}{b}$. 62. $4bd$. 63. $\frac{b-c}{a-b}$. 64. $\frac{4a+3c}{b+4}$.
65. $\frac{5ac+3bc}{d+1}$. 66. $\frac{ba}{ca-1}$. 67. $\frac{3a-6}{4}$.
68. $\frac{7-3b}{a-7}$. 69. $\frac{mr}{m+n+p}$. 70. $\sqrt{\frac{b-2a}{b}}$.
71. $\frac{abcd}{ab+ac+bc}$. 72. $\frac{(g+h)bdf}{adf+bcf+bde}$.
73. $\sqrt{\frac{a+b}{a-b}}$. 74. $b\sqrt{\left(1-\frac{2a}{bc}\right)}$. 75. $\frac{b-1}{a-7d}$
76. $\frac{70ab-3ac}{320c}$. 77. $\frac{8a}{25}$. 78. $\frac{3b}{3a-2b}$. 79. $\frac{ac}{b}$.
80. $\frac{a^2c+b^2a-c^2b-a-b-c}{ac+bc+ab-1}$. 81. $\frac{b^2-a^2}{4a-b}$.
82. $\frac{2a-b}{5b}$. 83. $\frac{ab}{a+b}$. 84. $\frac{(ad+bc)fh+(eh+fg)bd}{bdfhk}$.

Equations with Surds.

85. a^2. 86. 3. 87. 16. 88. 216. 89. 121.
90. 3. 91. $\frac{1}{4}(a^2 - 2a + 1)$. 92. 4. 93. 6.
94. 4. 95. 980. 96. $\frac{25}{16}a$. 97. $\frac{1}{1-a}$.
98. $\frac{1}{2}b\sqrt{(4a - b^2)}$. 99. $\frac{1}{9}$. 100. $\frac{a}{a + 2\sqrt{a}}$.
101. $\frac{a^2 + b^2}{2a}$. 102. $\left(\frac{ab}{a-b}\right)^2$. 103. $\frac{1}{9} \cdot \frac{b^2}{a}$.
104. $b\left(\frac{a+b}{a-b}\right)^2$. 105. $\frac{d-b}{a-c}$. 106. $\frac{9}{20}$. 107. $\frac{2}{3}$.
108. $5\frac{1}{7}$. 109. $1 - a$. 110. 200. 111. $\frac{25}{16}$.
112. $\frac{1}{3}\left(4 + \frac{a^2}{a-1}\right)^2$. 113. $\frac{2}{5}a\sqrt{5}$. 114. $\frac{a^2 - b^2}{3a}$.
115. 1. 116. $1\frac{1}{13}$. 117. $\frac{a}{2b}\left(b - 1\right)^2$. 118. $\frac{24}{25}$.
119. 8. 120. $\left(\frac{a^2 - 1}{a^2 + 3}\right)^{\frac{1}{2}}$. 121. $a\left(\frac{b+c}{b-c}\right)$.
122. $\frac{a}{(a^2 - 4)^{\frac{1}{2}}}$. 123. $\frac{(b-a)^2}{2a - b}$.

XIII.—P. 159.

1. $x = 3, y = 2$. 2. $x = 7, y = 12$. 3. $x = 8, y = 5$.
4. $x = 4, y = 5$. 5. $x = 2, y = 1$. 6. $x = 7, y = 10$.
7. $x = 5, y = 11$. 8. $x = 4, y = 12$. 9. $x = 1, y = 4$.
10. $x = \frac{cf - bh}{af - bd}$, $y = \frac{ah - dc}{af - bd}$.
11. $x = \frac{1}{2}(s + d)$, $y = \frac{1}{2}(s - d)$. 12. $x = 1, y = \frac{1}{2}$.
13. $x = \frac{bc}{a + b}$, $y = \frac{ac}{a + b}$. 14. $v = \frac{2}{3}, z = \frac{3}{2}$.
15. $x = \frac{nq - p}{n^2 - 1}$, $y = \frac{np - q}{n^2 - 1}$. 16. $x = 10, y = 8$.

17. $x = 57$, $y = 103$. 18. $x = \frac{p+2}{1+p^2}$, $y = \frac{1-2p}{1+p^2}$.

19. $x = .02$, $y = 2\ .\ 9$. 20. $x = \frac{4n-m}{6a}$, $y = \frac{2n+m}{6b}$.

21. $x = 3$, $y = 4$. 22. $x = 10$, $y = 20$.

23. $x = 16$, $y = 24$. 24. $x = \frac{m^2-n^2}{ma-nb}$, $y = \frac{m^2-n^2}{mb-na}$.

25. $x = \frac{bc-ad}{nb-md}$, $y = \frac{bc-ad}{mc-na}$. 26. $x = \frac{1}{10}$, $y = \frac{3}{8}$.

27. $x = \frac{3}{4} \cdot \frac{a}{b}$, $y = \frac{5}{6} \cdot \frac{a}{b}$. 28. $x = \frac{1}{2}$, $y = \frac{1}{8}$.

29. $x = \frac{a^2+b^2}{2(a+b)}$, $y = \frac{ab(a-b)}{a+b}$. 30. $x = 7$, $y = 5$.

31. $x = \frac{2b^2-6a^2+c}{3a}$, $y = \frac{3a^2-b^2+c}{3b}$.

32. $x = \frac{1}{2}(a+b)\sqrt{\frac{c}{ab}}$, $y = \frac{1}{2}(a-b)\sqrt{\frac{c}{ab}}$.

33. $x = \frac{abc(ab+ac-bc)}{a^2b^2+a^2c^2-b^2c^2}$, $y = \frac{abc(ac-ab-bc)}{a^2b^2+a^2c^2-b^2c^2}$.

34. $x = 7$, $y = 11$. 35. $x = 3$, $y = 5$. 36. $x = 3$, $y = 2$.

37. $x = \frac{ab}{a-b}$, $y = \frac{ab}{a+b}$. 38. $x = 7$, $y = 4$.

39. $x = -38.084$, $y = 56.814$. 40. $x = \frac{a}{bc}$, $y = \frac{a+2b}{c}$.

41. $x = 7$, $y = 5$. 42. $x = 7$, $y = 9$. 43. $x = 4$, $y = 3$.

44. $x = 7$, $y = 2$, 45. $x = 18$, $y = 24$. 46. $x = 16$, $y = 25$.

47. $x = 20$, $y = 15$, $z = 12$. 48. $x = 5$, $y = 11$, $z = 17$.

49. $x = 20$, $y = 8$, $z = 3$. 50. $x = 16$, $y = 7\frac{3}{4}$, $z = 5\frac{1}{2}$.

51. $x = 23$, $y = 6$, $z = 24$. 52. $x = 3$, $y = 4$, $z = 5$.

53. $v = 1$, $u = 2$, $w = 3$. 54. $v = 3$, $u = 7$, $w = 4$.

55. $x = 3$, $y = 7$, $z = 16$. 56. $x = 10$, $y = 1$, $z = 9$.

57. $x = \frac{2}{3}$, $y = -7$, $z = 36\frac{1}{3}$. 58. $x = 8$, $y = 5$, $z = 3$.

59. $x = 8$, $y = 17$, $z = 30$. 60. $x = 2$, $y = -3$, $z = 1$.

61. $x = 12$, $y = 10$, $z = 8$. 62. $x = 24$, $y = 9$, $z = 5$.

63. $x = \frac{2}{a+b-c}$, $y = \frac{2}{a+c-b}$, $z = \frac{2}{b+c-a}$.

64. $x = 2$, $y = 4$, $z = 6$. 65. $x = 12$, $y = 20$, $z = 30$.

66. $x = \frac{amp}{np+m(p+q)}$, $y = \frac{anp}{np+m(p+q)}$, $z = \frac{amq}{np+m(p+q)}$

67. $x = \frac{ce-bf}{ae-bd}$, $y = \frac{af-cd}{ae-bd}$, $z = \frac{a(el-fg)-d(bl-cg)}{h(ae-bd)}$

68. $x = \frac{1}{2}(a+b-c)$, $y = \frac{1}{2}(a+c-b)$, $z = \frac{1}{2}(b+c-a.)$

69. $x = 100$, $y = 60$, $z = -13$, $u = -50$.

70. $x = 7$, $y = 5$, $z = 4$. 71. $x = \frac{12}{7}$, $y = \frac{12}{5}$, $z = -12$.

72. $x = 1$, $y = 4$, $z = 27$.

73. $x = 5$, $y = 4$, $z = 3$, $u = 2$, $t = 1$.

74. $x = \frac{2abc}{ac+bc-ab}$, $y = \frac{2abc}{ab+bc-ac}$, $z = \frac{2abc}{ab+ac-bc}$.

75. $x = \frac{a^2}{\sqrt{(a^2+b^2+c^2)}}$, $y = \frac{b^2}{\sqrt{(a^2+b^2+c^2)}}$,

$z = \frac{c^2}{\sqrt{(a^2+b^2+c^2)}}$. 76. $x = 3$, $y = 7$, $z = 11$, $w = 20$.

XIV.—P. 175.

1. ± 4. 2. ± 6. 3. ± 7. 4. ± 4, 5. $\pm 3a^2$.
6. $\pm \frac{1}{2}\sqrt{ab}$. 7. 4, − 10. 8. 6, − 18. 9. 3, − 17.
10. 2, − 12. 11. 5, − 11. 12. 11, − 5.
13. 3, − 11. 14. 11, − 3. 15. 23, − 3.
16. 14, 6. 17. − 5, − 7. 18. − 12, − 4.
19. 5, − 3. 20. 10, 6. 21. − 10, − 3.
22. 20, − 30. 23. 22, 2. 24. − 11, − 5.
25. 6, $-4\frac{1}{2}$. 26. 4, $-1\frac{2}{3}$. 27. 4, $-2\frac{1}{3}$.
28. 6, $-5\frac{1}{3}$. 29. 2, $-\frac{1}{3}$. 30. $\frac{1}{2}$, $-\frac{4}{3}$.
31. $\frac{1}{2}$, − 2. 32. $\frac{1}{2}$, $\frac{1}{3}$. 33. 5, 8. 34. 7, − 6.
35. $\frac{1}{2}$, $-\frac{2}{3}$. 36. 12, − 7. 37. 8, − 15.
38. 11, − 10. 39. $\frac{1}{2}$, $-\frac{1}{3}$. 40. $9\frac{15}{17}$, − 11.
41. 70, 50. 42. 107, − 106. 43. 2, $\frac{1}{2}$.

44. $4 \pm \sqrt{30}$. 45. $\dfrac{3 \pm \sqrt{(-111)}}{4}$. 46. 4·732, 1·268.

47. $12, -\frac{3}{4}$. 48. $3, -8{\cdot}7$. 49. $4, 2\frac{1}{12}$.

50. $8 \pm \sqrt{601}$. 51. $1\frac{3}{4}, 1$. 52. 16, 36.

53. $3, \sqrt{-\frac{11}{2}}$. 54. $2, -3$. 55. $4, 7\frac{1}{8}$.

56. 4, 1. 57. $4, \sqrt[3]{\frac{1}{4}}$. 58. $32, -243$.

59. $9, 5 + (-5)^{\frac{4}{3}}$. 60. $3, \sqrt[3]{(-30)}$. 61. $3, \sqrt[3]{-9}$.

62. $\pm 5, \pm 7$. 63. $\dfrac{-3 \pm \sqrt{5}}{2}$. 64. $1 \pm \sqrt{(1-a^2)}$.

65. $\frac{3}{4}a, \frac{1}{2}a$. 66. $\dfrac{1 \pm \sqrt{10}}{3}$. 67. $2, -1$.

68. 49, 25. 69. $4, -21$. 70. $64, (-\frac{27}{8})^{\frac{6}{5}}$.

71. $2, -1$. 72. $\{\frac{1}{2}[a \pm \sqrt{(a^2-4)}]\}^{\frac{1}{n}}$.

73. $\dfrac{c \pm \sqrt{(c^2+4ac)}}{2(a+b)}$. 74. $1, \dfrac{-1 \pm \sqrt{-3}}{2}$.

75. $\left(\dfrac{-b \pm \sqrt{(4ac+b^2)}}{2c}\right)^{\frac{2}{3}}$. 76. $\dfrac{\pm 1 \pm \sqrt{-1}}{\sqrt{2}}$.

77. $\frac{1}{6}(7 \pm \sqrt{85}), \frac{1}{3}(1 \pm \sqrt{10})$.

78. $\pm \dfrac{1}{a}\left\{\sqrt{-1 \pm \sqrt{1-a^4} \pm \sqrt{2(1 \mp \sqrt{1-a^4})}}\right\}$.

79. $7, -1\frac{3}{7}$. 80. $4, -3\frac{2}{9}$. 81. $10, -29$.

82. $\pm 1, \pm 1 \pm \sqrt{-3}$. 83. $p \pm \sqrt{(p^2+q)}$.

84. $5a, -12a$. 85. $\dfrac{1}{4}, \dfrac{1}{9}$. 86. $\dfrac{a \pm \sqrt{(2ab-b^2)}}{2}$.

87. $4a \pm \sqrt{(16a^2+3b^2)}$. 88. $\sqrt[3]{\dfrac{1 \pm \sqrt{(4a+1)}}{2}}$.

89. $\dfrac{1}{2}b \pm \sqrt{\dfrac{4a^3-b^3}{12b}}$. 90. $-a, -b$. 91. $\dfrac{n}{q}, -\dfrac{p}{m}$.

92. $\dfrac{a+b}{a-b}, \dfrac{a-b}{a+b}$. 93. $-1, \dfrac{3}{5}$. 94. $7\dfrac{22}{113}, 2\dfrac{1}{2}$.

95. $\frac{2}{3}, \frac{3}{10}$. 96. $\pm 6; \pm \sqrt{51}$. 97. $1, 1 \pm 2\sqrt{15}$.

98. $6a, -\frac{2}{3}a$. 99. $\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$.

100. $6, -1, \frac{5 \pm 3\sqrt{(-3)}}{2}$. 101. $\frac{a}{2} \pm \frac{1}{4} \cdot a\sqrt{2}$.

102. $\pm 13, \pm 2\sqrt{-1}$. 103. $\frac{3a}{32}(3 \pm \sqrt{21})$.

104. $\pm\sqrt{2}, \pm\sqrt{3}$. 105. $4, \frac{1}{2}, \pm\frac{5 \pm \sqrt{1329}}{4}$.

106. $\frac{1}{2}(1 \pm \sqrt{5})$. 107. $\frac{(1 \pm \sqrt{5})^n - 2^n}{(1 \pm \sqrt{5})^n + 2^n}$.

108. $\pm\sqrt{\frac{1 \pm \sqrt{5}}{2}}$. 109. $\frac{3 \pm \sqrt{13}}{2}, 9 \pm 4\sqrt{7}$.

110. $\left(\frac{a \pm b}{a \mp b}\right)^{\frac{2pq}{q-p}}$.

XV.—P. 184.

1. $x = 4, y = 2$. 2. $x = \pm 15, y = \pm 6$. 3 $x = 5, y = 3$.

4. $x = 25, 9$; $y = 9, 25$. 5. $x = 6, y = 3$.

6. $x = 4, 2, 3 \pm \sqrt{21}$; $y = 2, 4, 3 \mp \sqrt{21}$.

7. $x = 5, \frac{17}{10}$; $y = 3, -\frac{3}{10}$. 8. $x = 2, 5, y = 6, 3$.

9. $x = 2, -46, y = 3, 15$.

10. $x = \frac{a \pm \sqrt{(a^2 - 4b)}}{2}, y = \frac{a \mp \sqrt{(a^2 - 4b)}}{2}$.

11. $x = \frac{a \pm \sqrt{(2c - a^2)}}{2}, y = \frac{a \mp \sqrt{(2c - a^2)}}{2}$.

12. $x = \pm 2, \mp \frac{5}{\sqrt{31}}$; $y = \pm 3, \pm \frac{6}{\sqrt{31}}$.

13. $x = \pm 3, \pm \frac{8}{\sqrt{6}}$; $y = \pm 1, \pm \frac{1}{\sqrt{6}}$.

14. $x = \pm 9, 8\sqrt{2}, y = \pm 7, \sqrt{2}$.

15. $x = b + b\sqrt{\frac{a - 2b}{a + 6b}}, y = b - b\sqrt{\frac{a - 2b}{a + 6b}}$.

16. $x = 3, 2, y = 2, 3$. 17. $x = 11, 9, y = 9, 11$.

18. $x = 9, 15, y = 15, 9$. 19. $x = 5, y = 2$.

20. $x = 6, y = 7$. 21. $x = 5, 3, y = 3, 5$.

22. $x = 12, 3, y = 6, z = 3, 12.$ 23. $x = 3, y = 7.$

24. $x = \dfrac{4a + 1 \mp \sqrt{(8a + 1)}}{2}, y = \dfrac{-1 \pm \sqrt{(8a + 1)}}{2}.$

25. $x = \pm 2, \pm \frac{23}{58}\sqrt{29}, y = \pm 1, \pm \frac{9}{58}\sqrt{29}.$

26. $$x = \frac{a}{2} \pm \sqrt{\left(-\frac{a^2}{4} \pm \sqrt{\frac{a^5 + 4b}{20a}}\right)}, y = \frac{a}{2} \mp \sqrt{\left(-\frac{a^2}{4} \mp \sqrt{\frac{a^5 + 4b}{20a}}\right)}.$$

27. $$x = \frac{a}{2} \pm \sqrt{\left(-\frac{3a^2}{4} \pm \sqrt{\frac{a^4 + b}{4}}\right)}, y = \frac{a}{2} \mp \sqrt{\left(-\frac{3a^2}{4} \mp \sqrt{\left(\frac{a^4}{4} + b\right)}\right)}.$$

28. $x = \pm 7, \pm 4$; $y = \pm 4, \pm 7.$ 29. $x = 17, 1$; $y = 1, 17.$

30. $x = 5, -2$; $y = 2, -5.$ 31. $x = 9, 6\frac{1}{4}, y = 4, 6\frac{1}{4}.$

32. $x = 3, y = 1.$ 33. $x = 5, \sqrt{\frac{45}{2}}, y = 4, \sqrt{\frac{37}{2}}.$

34. $x = \dfrac{57 \pm \sqrt{105},}{2}$ $y = \dfrac{3 \pm \sqrt{105}.}{6}$ 35. $x = \pm \frac{5}{3}, y = \pm \frac{13}{3}.$

XVI.—P. 191.

1. $q - \dfrac{m + n}{p}$, and $q - \dfrac{m - n}{p}.$ 2. $x - 4 + b$, and $x - 5 + a.$

3. $a - 11, a + 54.$ 4. $x - 80, x + 75.$ 5. $x^2 - x = 90.$

6. $x^2 - \frac{5}{6}x + \frac{1}{6}.$ 7. $a + 7, a - 4.$

8. This reduces to $x^2 - 2ax = b^2 - a^2$; roots are $a + b$ and $a - b$.

9. $x^2 - \frac{73}{12}x = \frac{77}{12}.$ 10. $x - 136, x + 25.$ 11. 7, 11.

12. $\frac{1}{10}, \frac{1}{11}.$ 13. 18, -2. 14. 16, -20.

XVII.—P. 198.

1. 24. 2. 4. 3. $\frac{1}{2}(a - b).$ 4. $\frac{1}{2}(a + b).$

5. $\dfrac{abc}{a + b}.$ 6. 16 and 25. 7. 34, 22. 8. £82, £68.

9. 8. 10. £89, £98, £113. 11. 80.

12. 24 feet. 13. 33 feet 4 inches. 14. 72 lbs.

15. In six hours, and 216 miles from London.

16. $232\frac{4}{11}$ miles from London. 17. 34, 23, 18

18. $\frac{a}{m+1}, \frac{ma}{m+1}.$

19. $\frac{a}{1+m+n}, \frac{ma}{1+m+n}, \frac{na}{1+m+n}.$

20. 385, 427. 21. The woman's 15, the man's 45.

22. 22, 7, 12. 23. 34 of brandy, and 43 of rum.

24. 20 days. 25. 22, 12.

26. He worked 20 days, and was idle 10.

27. $\frac{ab}{a+b}$; $\frac{abc}{ab+ac+bc}$; $\frac{abcd}{abc+abd+bdc+adc}$; $2\frac{26}{47}$; $6\frac{2}{3}$; $1\frac{31}{89}$. 28. 72. 29. $5\frac{5}{11}$ min. past one.

30. £$116\frac{2}{3}$, £$93\frac{1}{3}$. 31. 1810. 32. 14 and 42.

33. $\frac{5}{6}$ mile per hour; 4 hours; 6 hours.

34. 420 copper, 85 tin. 35. £10 : 8s.

36. The one £500, and the other £700.

37. £3 and £2. 38. $\frac{5}{18}$. 39. 50 of each.

40. The man in $21\frac{3}{7}$ days, the woman in 50 days.

41. 20 of rye, and 52 of wheat.

42. 45 turkeys, and 60 geese. 43. 71. 44. 42.

45. 120s. 46. 40 and 24. 47. 6, 13, 25.

48. £20; (or -120). 49. 26, 38. 50. 18, £4.

51. 9, 12, 15. 52. 30 by 25.

53. 10 and 9 miles per hour. 54. 21, 25. 55. 7.

56. 20, 15. 57. 12, 6.

58. 4 m. per hr.; 3 m. per hr.; dist. $3\frac{1}{2}$ m.

59. $\frac{1}{2}$. 60. 10, 6, 8. 61. 7. 62. 23, 28.

63. £192, £224. 64. 25 miles from London.

65. 4·9 seconds in 24 hours.

66. $\frac{2abc}{ac+bc-ab}$ days, $\frac{2abc}{ab+bc-ac}$ days, $\frac{2abc}{ab+ac-bc}$ days; $14\frac{34}{49}$, $17\frac{23}{41}$, $23\frac{7}{31}$. 67. 12. 68. 40 hrs., 60 hrs.

69. 48 miles, 40 minutes; $3\left(2-\frac{b'}{b}\right)a, 3\cdot\frac{b-b'}{bb'}\left(2-\frac{b'}{b}\right)a.$

70. $15\frac{5}{8}$ miles from Edinburgh.

XVIII.—P. 214.

1. $\frac{a}{b}\left(1 - \frac{c}{b}x + \frac{c^2}{b^2}x^2 - \frac{c^3}{b^3}x^3 + \&c.\right)$

2. $\frac{a}{a'} - \left(\frac{b'}{a'}A - \frac{b}{a'}\right)x - \left(\frac{b'}{a'}B - \frac{c'}{a'}A\right)x^2 - \left(\frac{b'}{a'}C + \frac{c'}{a'}B\right)x^3$ &c.

3. $1 + 2ax + (4a^2 - 1)x^2 + (8a^3 - 4a)x^3$ &c.

4. $1 + \frac{x^2}{2} - \frac{x^4}{2\,.\,4} + \frac{1\,.\,3x^6}{2\,.\,4\,.\,6} - \frac{1\,.\,3\,.\,5x^8}{2\,.\,4\,.\,6\,.\,8} + \&c.$

5. $1 - \frac{x}{2} - \frac{x^2}{2\,.\,4} - \frac{3x^3}{2\,.\,4\,.\,6} - \frac{3\,.\,5x^4}{2\,.\,4\,.\,6\,.\,8} - \frac{3\,.\,5\,.\,7x^5}{2\,.\,4\,.\,6\,.\,8\,.\,10}.$

6. $a - 2ax + 3ax^2 - 4ax^3$ &c.

7. $\frac{3}{5} - \frac{11}{5^2}x + \frac{7\,.\,11}{5^3}x^2 - \frac{7^2\,.\,11}{5^4}x^3 + \frac{7^3\,.\,11}{5^5}x^4 - \&c.$

8. $1 + 3x + 4x^2 + 7x^3 + 11x^4 + 18x^5 + 29x^6$ &c.

9. $1 - x + x^4 - x^5 + x^8 - x^9 + \&c.$ or $(1 - x)(1 + x^4 + x^8 + x^{12}$ &c.)

10. $\frac{4}{3(x-1)}, -\frac{1}{3(x+2)}.$ 11. $\frac{-4}{x-3}, \frac{5}{x-4}.$

12. $\frac{4}{3(x-2)}, -\frac{1}{2(x-1)}, \frac{1}{6(x+1)}.$

13. $\frac{1}{(a-b)(a-c)(x+a)}, -\frac{1}{(a-b)(b-c)(x+b)}, \frac{1}{(a-c)(b-c)(x+c)}.$

14. $\frac{1}{4a^3(x-a)} - \frac{1}{4a^3(x+a)} - \frac{1}{2a^2(x^2+a^2)}$

15. $\frac{1}{3(x+1)}, -\frac{1}{2x}, -\frac{1}{6(x-2)}.$ 16. $4x + 5$, $3x - 5$.

17. $2x + y - 3$, $x - 11y + 1$. 18. $x + ax^3 + (3a^2 - b)x^5 + \&c.$

19. $x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$ &c. 20. $\frac{x}{3} + \frac{x^3}{3^4} + \frac{x^5}{3^6} + \&c.$

21. $\frac{1}{2}x - \frac{3}{16}x^3 + \frac{12}{128}x^5 - \frac{152}{1024}x^7$ &c.

22. $x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x+1)^4}{4}$ &c.

XIX and XX.—P. 232.

1. $5 : 8$. 2. $13 : 11$. 3. Diminished. 4. Increased.
5. $a + b : a - b$. 6. $b : a$. 7. $18y : 35x$. 8. $3 : 2$.
9. $a^2 : x^2$. 10. $a^3 : x^3$. 11. $12 : 11 > 14 : 13 > 16 : 15$.
12. $a + 4 : \frac{1}{2}a + 5$ is the greater. 13. $3 \times 6 \times 4 = 72$.
14. $x = 36$. 15. $728\frac{1}{3} : 728$ nearly. 16. $\frac{1}{x} : \frac{1}{a}$.
18. $x = 30$. 20. $175 : 9$.
21. $1001 : 1000$; $1501 : 1500$.
22. Of equality; of greater inequality.
23. $y = 6x$; $y = 5x$; $y = (a + b)x$; $y = 5x$.
24. $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$, $m = \frac{b^2}{a^2}$. 25. $y = 12$.
26. $y = 6x + \frac{4}{x^2}$. 27. $s = 16\frac{1}{2} \times t^2$. 28. $x = \frac{a}{c^{mn}} \cdot z^{mn}$.
29. If when $x = 1, 2, 3$, $y = 6, 11, 18$; then $y = 3 + 2x + x^2$.
30. $y = \frac{5}{6}(x + \sqrt{x})$.

XXI.—P. 242.

1. 819. 2. 1176. 3. 442. 4. 55, 171, 300.
5. 100, 324, 576. 6. 110, 342, 600. 7. 27.
8. 330, 0. 9. 27600. 10. 0. 11. $\frac{n(n + 1)}{2}$.
12. n^2. 13. $n(n + 1)$. 14. $n(8 + n)$.
15. 3069, 196, 605. 16. 511, 16, 383. 17. 436905.
18. 118096. 19. 265720. 20. -132860.
21. $\frac{7}{15}$. 22. $13\frac{1}{2}$. 23. 0. 24. 142.
25. 1900. 26. $162\frac{1}{2}$. 27. 5050, 505.
28. -9801. 29. $\frac{174075}{59049}$. 30. $\frac{5}{7}$. 31. 16.
32. $\frac{3}{2}$, 2. 33. $\frac{9}{4}$. 34. $\frac{8}{5}$. 35. $10\frac{1}{8}$.
36. $\frac{1}{\sqrt{2}} \cdot \sqrt{\frac{2^n - 1}{\sqrt{2} - 1}}$, $3\sqrt{\frac{3}{2}}$. 37 $\frac{3}{10}$. 38. 10.

39. 70. 40. $\frac{1}{x+1}$. 41. $\frac{x}{a+b}$. 42. $\frac{n}{6}(79-n)$.

43. $\frac{n}{12}(13-7n)$. 44. $\frac{1}{2}(n-1)$.

45. $\frac{a+b}{(a+b)^{n-2}}\cdot\frac{(a-b)^n-1}{a-b-1}$. 46. 0. 47. 416.

48. 416. 49. 416. 50. 18, 915, $n=65$.

51. 13, 22, 31, 40.

52. 243, 293, 343, 393; $-2, -6, -10, -14$.

53. 207, 297, 387; $\frac{3}{8}$, $\frac{5}{12}$, $\frac{11}{24}$.

54. 15, 45, 135; $r=3$; $\pm 2 . 5, 5, \pm 10$; $\frac{1}{2}\sqrt{\frac{2}{3}}$, $\frac{1}{3}$, $\frac{1}{3}\sqrt{\frac{2}{3}}$.

55. 117, 351, 1053. 56. 27. 57. $7\frac{1}{2}$, 3. 58. $\frac{4}{5}$.

59. $\frac{1}{3}$, 1, 3, 9. 60. 35, 175, 875; 21, 147, 1029.

61. 72, 63; $\frac{6}{7}$, $\frac{3}{4}$, 8, 12.

62. $\frac{12}{5}$, 3, 4, 6; $\frac{6}{5}$, $\frac{3}{4}$, $\frac{6}{11}$, $\frac{3}{7}$, $\frac{6}{17}$, $\frac{3}{10}$.

63. 441·216 . . . $\pm\frac{8}{9}$; $\pm(a+x)$.

64. $\frac{13}{16}$, $\frac{5}{8}$, $\frac{7}{16}$, $\frac{1}{4}$, $\frac{1}{16}$, $-\frac{1}{8}$, $-\frac{5}{16}$.

65. $3\frac{5}{24}$; $\frac{n-2}{n}$. 66. 6 or 12.

67. 4 or -10, 4, 21 or -27. 68. 3. 69. 3. 70. $\frac{20}{13}$.

71. 44. 72. 14. 73. 19 or 18. 74. 26. 75. 3.

76. 24. 77. $\frac{3}{2}$, 18, 12. 78. $\frac{41}{333}$, $\frac{125}{333}$, $\frac{86}{99}$.

79. 7 or 13. 80. $\frac{1}{2048}$, $15\frac{2047}{2048}$.

81. $\frac{4}{27}$, $\frac{5}{36}$, $\frac{25}{44}$, $\frac{145}{222}$. 82. 19. 83. 60.

. . . 88. 4 and 2. . . . 91. $\frac{6}{5}$, $\frac{3}{2}$, 2, 3, 6, $\frac{1}{6}$, -6.

92. $\frac{1}{14}\cdot\frac{5^n \mp 2^n}{5^{n-2}}$; $\frac{1}{x^{\frac{3n-3}{2}}}\cdot\frac{x^{\frac{3n}{2}} \mp a^n}{x^{\frac{3}{2}}+a}$.

93. $n=14$; series is 1, 3, 5, &c.

94. $4+3\sqrt{2}$; $a^p\cdot\frac{a^{nq}-1}{a^q-1}$.

. . . 96. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ &c., or $\frac{1}{2}$, $\frac{7}{4}$, $-\frac{7}{6}$.

XXII.—P. 250.

1. 7, 21, 63, 189. 2. £120, £60, £30. 3. 2, 4, 8.
4. 2, 4, 8, 16. 5. 5, 7, 9. 6. 3, 6, 12.
7. 4680 times. 8. 2, 4, 8. 9. 6, 8, 10.
10. 3, 6, 12. 11. 11, 9, 7, 5. 12. £110 : 9s.
13. 7999 acres.
14. 8 days; 198 days; 19,701 miles, and 3960 miles.
15. 3 days, 36 miles. 16. 8 days. 17. 6, 4, 3.
18. 10, 40. 19. 6, 8, 12. 20. £108, £144, £192, £256.
21. £64000. 22. 143. 23. 234.
24. 9; if salient, 16. 25. 400 miles.
26. $\frac{7}{10}$, $2\frac{2}{10}$, $3\frac{7}{10}$, $5\frac{2}{10}$, $6\frac{7}{10}$; $1\frac{5}{8}$, $3\frac{5}{8}$, $5\frac{5}{8}$, $7\frac{5}{8}$.
27. £1,816,033 : 10 : 10. 28. $19\frac{1}{2}$ miles, 640 feet.
29. Radius of order double that of inner sphere.
30. Value of ruby $= \frac{cn^{\frac{3}{2}}}{(m+1)b^{\frac{3}{2}}}$.

XXIII.—P. 274.

1. 32. 2. 4.
3. $3 \cdot 1^3 + 5 \cdot 2^3 + 7 \cdot 3^3 + 9 \cdot 4^3$ &c., or 3, 40, 189, 576, 1375, 2808, 5145, &c.; $d_4 = 48$. 4. 328000, 1775480.
5. 2, 6, 12, 20, 30, &c.; 420.
6. $1 \cdot 3^2 + 2 \cdot 5^2 + 3 \cdot 7^2 +$ &c.; 24, 24, 24, &c.; 100th term 2030100; $s = 103368450$.
7. $1 + 2 + 5 + 13 + 33 + 76$, &c.; 112103; 206242.
8. 256. 9. The one is double the other.
10. $1^2 \cdot 8 + 2^2 \cdot 11 + 3^2 \cdot 14 + 4^2 \cdot 17 +$ &c.; first terms of difference, 36, 46, 18, 0; 987554.
11. $\frac{n(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3 \cdot 4}$. 12. 171700.
13. 3600, n^2.
14. $\frac{1}{6}n(n+1)(2n+1)$; see Arts. 228, 230, $\frac{1}{3}n(4n^2-1)$.
15. $\frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$; $\frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$.
16. $\frac{1}{8}$. 17. $\frac{1}{6}$. 18. $n^2(2n^2+3) + \frac{1}{3}n(16n^2-4)$.

19. $\dfrac{n(n+1)(n+2)}{1\,.\,2}$; $\dfrac{n(n+1)(n+2)(n+3)}{4}$.

20. $\dfrac{7}{180}=\dfrac{7}{2\,.\,3\,.\,5\,.\,6}$.

21. $\dfrac{n(4n^2+18n+23)}{3\,.\,1\,.\,3\,.\,5(2n+1)(2n+3)(2n+5)}$; $\dfrac{1}{90}$.

22. $\frac{1}{12}$. 23. $\frac{1}{32}$. 24. $\frac{1}{288}$. 25. $\frac{3}{2}$. 26. $\frac{1}{72}$.

27. $\frac{1}{6}$. 28. $\frac{5}{6}$. 29. $\dfrac{1}{ax}$.

30. $\dfrac{a-ar^n}{1-r}-\dfrac{nar^n}{1-r}$; $\dfrac{a}{(1-r)^2}$. 31. $\dfrac{1+x}{(1-x)^2}$; scale, $2x-x^2$.

32. $\dfrac{1}{1-2x-3x^2}$; scale, $2x+3x^2$.

33. $\dfrac{1}{(1-x)^2}$; scale, $2x-x^2$. 34. $\dfrac{1}{(1+x)^2}$; scale, $-2x-x^2$.

35. $\dfrac{4-11x}{1-5x+6x^2}$. 36. $\dfrac{1+5x}{1-x-6x^2}$. 37. $\dfrac{1}{1-2x+x^2}$.

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XXIV.—P. 289.

1. $x=\dfrac{\log.\ b}{\log.\ a}$. 2. $x=\dfrac{\log.\ c}{b\log.\ a}$. 3. $x=1{\cdot}537$.

4. 1·397940; $\bar{2}$·096910. 5. $x=17.917$.

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9. $-$·991. 10. 7; 9.

11. Log. $x=\dfrac{q}{q-p}(\log.\ p-\log.\ q)$, log. $y=\dfrac{p}{q-p}(\log.\ p-\log.\ q)$.

.... 13. 1·9190781. 14. 2·0128372.

XXV.—P. 296.

1. 132 ; 1320 ; 11880. 2. 210 ; 4200 ; 8400.
3. 6720 ; 40320.
4. 3628800 ; 2520, 1120 ; 34650 ; 454053600.
5. 2520. 6. 276 ; 2024 ; 10626.
7. 276 ; 2024 ; 10626.
8. 2496144 ; 2704156 ; 2496144.
9. 9, 36, 84, 126, 126, 84, 36, 9.
10. 635013559600. 11. 120. 12. 7.
13. 12650 ; 2024. 14. 190 ; 171. 15. 720.
16. 46200. 17. 400400. 18. $\frac{n(n-1)(n-2)}{1.2.3}$.
19. 2800. 20. 4. 21. 63. 22. 75600.
23. 52. 24. 14.

XXVI.—P. 308.

1. $1+\frac{5x}{2}+\frac{5x^2}{2}+\frac{5x^3}{4}+\frac{5x^4}{16}+\frac{x^5}{32}$; $1+x+\frac{3}{4}x^2+\frac{1}{2}x^3+\frac{5}{16}x^4$ + &c. ; $a^3+6a^2b+12ab^2+8b^3-3a^2c-12abc-12b^2c+3ac^2+6bc^2-c^3$; $1+n\left(\frac{2x}{a-x}\right)+\frac{n(n-1)}{1.2}\left(\frac{2x}{a-x}\right)^2$ + &c.

2. The second term ; the first term ; the second term.

3. The 23d. 4. $924a^{12}x^{12}$, $1716a^7x^6$, $1716a^6x^7$.

5. $(-1)^{r-2}\frac{1.2.5\,....\,(3r-7)}{1.2.3\,....\,(r-1)}a^{\frac{4-3r}{3}}.\left(\frac{x}{3}\right)^{r-1}$; $61236a^{15}x^6$. 6. $\frac{1.3.5\,....\,(2n-1)}{\underline{|n}}2^nx^n$.

XXVII.—P. 317.

1. $\frac{3}{11}$. 2. $\frac{4}{33}$.

3. The prob. $=x=\frac{\log. 2}{\log. 36-\log. 35}$, or 24 · 6.

4. $\frac{5}{324}$. 5. $\frac{1}{7}$. 6. $\frac{10}{21}$. 7. $\frac{1}{120}$.
8. $(\frac{3}{10})^3 = \frac{27}{1000}$. 9. $\frac{11}{18}$. 10. 2 to 1.

XXVIII.—P. 324.

1. $\frac{ab}{a+b}$. 2. Side $= d(1+\sqrt{2})$.

3. Side $= \frac{b^2 - a^2}{2b}$; Hyp. $= \frac{a^2+b^2}{2b}$. 4. Side $= \frac{4s \pm s\sqrt{2}}{14}$.

5. $m\sqrt{\frac{a^2-b^2}{m^2-n^2}}$, $n\sqrt{\frac{a^2-b^2}{m^2-n^2}}$. 6. $\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} - b^2\right)}$.

7. Dist. of point from wall $= \frac{a^2+b^2-c^2}{2a}$.

8. Hyp. $= \frac{b \pm \sqrt{(b^2+4a^2)}}{2}$; side $= \sqrt{\frac{b^2 \pm b\sqrt{(b^2+4a^2)}}{2}}$.

9. Base $= \frac{1}{2}d \pm \frac{1}{2}\sqrt{(2a^2-d^2)}$; perp. $= -\frac{1}{2}d \pm \frac{1}{2}\sqrt{(2a^2-d^2)}$.

10. $x = \frac{a \pm \sqrt{(a^2-4bc)}}{2}$, x and $a-x$ being the segments made by a diameter.

11. Diag. $= a+b \pm \sqrt{2ab}$, sides, $b \pm \sqrt{2ab}$, and $a \pm \sqrt{2ab}$.

12. Base $\frac{a^2-b^2}{d}$.

13. Segt. of base, $= \frac{m^2a}{m^2-n^2} \pm \sqrt{\left(\frac{mna}{m^2-n^2}\right)^2 - b^2}$;

intercept of base $= \sqrt{\frac{4d^2(a^2+b^2)-d^4}{16a^2-4d^2}}$.

14. $\frac{b\sqrt{(d^2-a^2)}+a\sqrt{(d^2-b^2)}}{d}$.

15. Sides of inscribed and circumscribed pentagons $= \frac{1}{2}r\sqrt{(10-2\sqrt{5})}$, and $2r\sqrt{(5-2\sqrt{5})}$. Sides of inscribed and circumscribed decagons $= \frac{1}{2}r(\sqrt{5}-1)$ and $2r\sqrt{\frac{5-2\sqrt{5}}{5}}$.

16. Segt. of hyp. made by perp. $= \frac{d(d-c)}{d+c}$.

ERRATA.

The following errata occur in a portion of the impression.

P. 17, Ex. 60, for a, read x.
P. 63, Ex. 15, for $31x$, read $31x^2$.
P. 80, Ex. 42, for $20a^{n+}$, read $20a^{n+2}$.
P. 80. Ex. 56, should be $4ax^3 - 6b^2x^3 + 2x^3 - 10bx^2 \times \frac{1}{2x^3}$.
P. 82, Ex. 104, sign *minus* omitted in numerator.
P. 117, Ex. 3, for $\frac{1}{4}a^2$, read $\frac{1}{4}x^2$.
P. 117, Ex. 5, for $6ax$, read $6ax^3$.
P. 117, Ex. 12, for $2rsx^2$, read $2rsx^5$.
P. 117, Ex. 14, for $+125$, read -125.
P. 191, Ex. 8, for $acx + bcx^2$, read $2acx + 4cx^3$.

The note referred to on page 1 has not been inserted.

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